General Topology lecture notes

Thomas Baird

Winter 2011

Contents

1 Introduction

Topology is the study of *continuity*. We begin with a familiar definition of continuity.

Definition 1. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if $\forall x \in \mathbb{R}$ and $\forall \epsilon > 0$, $\exists\ \delta > 0$ such that $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$.

Here we use notation, $B_\delta(x) = \{x' \in \mathbb{R} | |x - x'| < \delta\}$, called the open interval (or ball) of radius δ centred at x.

The quantity $|x - x'|$ should be thought of as the "distance" between x and x'. This definition of continuity generalizes to sets equipped with an appropriate notion of distance.

Definition 2. Let X be a set. A *metric* on X is a function

$$d : X \times X \to \mathbb{R}_{\geq 0}$$

called the *distance* or *metric* function, satisfying

1. $d(x, x') = 0 \Leftrightarrow x = x'$ (d separates points)
2. $d(x, x') = d(x', x)$ (d is symmetric)
3. $d(x, x'') \leq d(x, x') + d(x', x'')$ (the triangle inequality)

A set X equipped with a metric d is called a *metric space*.

Example 1. $X = \mathbb{R}^n$ with the Euclidean metric $d(\vec{a}, \vec{b}) = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$.

Example 2. $X = \mathbb{R}^n$ with the taxi-cab metric $d(\vec{a}, \vec{b}) = \sum_{i=1}^n |a_i - b_i|$.

Example 3. $X = Cont([0,1], \mathbb{R})$ the set of real valued continuous functions on the interval, with L^p-metric $d(f,g) = (\int_0^1 |f-g|^p)^{1/p}$, where $p \geq 1$ is a real number.

Example 4. $X = \mathbb{Z}$ with p-adic metric $d(m,n) = p^{-k}$ where p is a prime number and p^k is the largest power of p dividing $m-n$.

Definition 3 (version I)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is *continuous* if $\forall x \in X$, $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$f(B_\delta(x)) \subset B_\epsilon(f(x)).$$

where $B_\delta(x) = \{x' \in X | d_X(x', x) < \delta\}$ and $B_\epsilon(y) = \{y' \in Y | d_Y(y', y) < \epsilon\}$.

This is where topology stood before the advent of topological spaces.

Definition 4. Let (X, d) be a metric space. A subset $U \subset X$ is called *open* if $\forall x \in U$, $\exists \epsilon > 0$ such that $B_\epsilon(x) \subset U$.

Example 5. In $\mathbb{R}^n$ the Euclidean and taxi-cab metrics give rise to the same class of open sets.

Definition 5 (version II)**.** A map $f : X \to Y$ between metric spaces is *continuous* if pre-images of open sets are open. I.e.

$$U \subseteq Y \text{ is open } \Rightarrow f^{-1}(U) := \{x \in X | f(x) \in U\} \subseteq X \text{ is open}$$

Exercise 1. Show that these two definitions of continuity are equivalent.

Observe that the second definition of continuity makes no explicit mention of the metric. Once we have determined which sets are open, the metric can be discarded. Thus this definition makes sense for the more general class of *topological spaces* defined below.

Definition 6. A *topological space* (X, τ) is a set X and a collection τ of subsets of X, called the *open sets*, satisfying the following conditions:

i) $\emptyset$ and X are open,
ii) Any union of open sets is open,
iii) Any finite intersection of open sets is open.

Exercise 2. Show that the open sets of a metric space determine a topology. This is called the *metric topology.*

What do we gain from this abstraction?

I. Different metrics may induce the same topology

- Can choose a metric suited to particular purpose
- Metrics may be complicated, while the topology may be simple
- Can study families of metrics on a fixed topological space

II. Some interesting topologies do not come from metrics

- Zariski topology on algebraic varieties (algebra and geometry)
- The weak topology on Hilbert space (analysis)
- Any interesting topology on a finite set (combinatorics)

2 Set Theory

We adopt a naive point of view on set theory, and assume that what is meant by a set is intuitively clear.

Let X be a set. A *subset* $A \subseteq X$ is a set whose elements all belong to X.

Let I be an *index set* (which may be finite, infinite or uncountable). For each $i \in I$, let $A_i \subseteq X$. Define the *union* to be

$$\bigcup_{i \in I} A_i := \{x \in X | x \in A_i \text{ for some } i \in I\}$$

and the *intersection*

$$\bigcap_{i \in I} A_i := \{x \in X | x \in A_i \text{ for all } i \in I\}$$

Proposition 2.1. *Let I and A_i be as above, and let $B \subseteq X$. Then intersections distribute over unions and vice-versa. I.e.*

$$\text{i)} \qquad B \cap \big(\bigcup_{i \in I} A_i\big) = \bigcup_{i \in I}(B \cap A_i)$$

$$\text{ii)} \qquad B \cup \big(\bigcap_{i \in I} A_i\big) = \bigcap_{i \in I}(B \cup A_i)$$

Proof. i)

$$\begin{aligned} B \cap \big(\bigcup_{i \in I} A_i\big) &= \{x \in X | x \in B \text{ and } x \in A_i \text{ for some } i \in I\} \\ &= \{x \in X | \text{ for some } i \in I, x \in A_i \text{ and } x \in B\} \\ &= \bigcup_{i \in I}(B \cap A_i) \end{aligned}$$

ii) is similar. □

The complement $A \subset X$, denoted A^c is defined

$$A^c := X \setminus A = \{x \in X | x \notin A\}$$

Proposition 2.2 (De Morgan's Laws)**.** *Let I and A_i be as above. The following identities hold.*

$$\text{i)} \qquad \big(\bigcup_{i \in I} A_i\big)^c = \bigcap_{i \in I} A_i^c$$

$$\text{ii)} \qquad \big(\bigcap_{i \in I} A_i\big)^c = \bigcup_{i \in I} A_i^c$$

Proof. i)

$$\begin{aligned}\big(\bigcup_{i \in I} A_i\big)^c = \{x \in X | x \notin \bigcup_{i \in I} A_i\} &= \{x \in X | x \notin A_i \text{ for all } i \in I\} \\ &= \{x \in X | x \in A_i^c \text{ for all } i \in I\} = \bigcap_{i \in I} A_i^c\end{aligned}$$

ii) is equivalent to i) (Prove this). □

Let $f : X \to Y$ be a map of sets. The *preimage* of a set $A \subseteq Y$ is defined

$$f^{-1}(A) := \{x \in X | f(x) \in A\}.$$

Proposition 2.3. *Preimages commute with unions and intersections.*

$$f^{-1}\big(\bigcup_{i \in I} A_i\big) = \bigcup_{i \in I} f^{-1}(A_i)$$

$$f^{-1}\big(\bigcap_{i \in I} A_i\big) = \bigcap_{i \in I} f^{-1}(A_i)$$

Proof. i)

$$\begin{aligned}f^{-1}\big(\bigcup_{i \in I} A_i\big) = \{x \in X | f(x) \in \bigcup_{i \in I} A_i\} &= \{x \in X | f(x) \in A_i \text{ for some } i \in I\} \\ \{x \in X | x \in f^{-1} A_i \text{ for some } i \in I\} &= \bigcup_{i \in I} f^{-1}(A_i)\end{aligned}$$

ii) is similar

□

We remark that the same property does not hold true for forward images. For example consider the case where X consists of two points, Y is a single point and $f : X \to Y$ is the (unique) map. If A_1 and A_2 are the two points in X, then $f(A_1 \cap A_2) = \emptyset$ but $f(A_1) \cap f(A_2) = Y$.

2.1 Order Relations

A relation on a set X a subset of $R \subset X \times X$. We use notation xRy to denote $(x, y) \in R$ and $x \not R y$ (this is meant to be a bar through R) to denote $(x, y) \notin R$. We will study two general types of relations : order relations and equivalence relations.

Definition 7. A *partial order* on a set X is a relation $<$ satifying

i) $x < y$ and $y < z$ implies $x < z$.

ii) $x \not< x$ for all $x \in X$

Warning: what we have called a partial order is sometimes called a strict partial order.

Proposition 2.4. *$x < y$ implies $y \not< x$.*

Proof. If $x < y$ and $y < x$, then by ii) $x < x$ contradicting i). □

Example 6. The real numbers $\mathbb{R}$ and the integers $\mathbb{Z}$ with the usual $<$.

Example 7. For any set X, the power set $\mathcal{P}(X)$ is the set of all subsets of X. Then $\mathcal{P}(X)$ is partially ordered by strict inclusion: $A < B$ if $A \subseteq B$ and $A \neq B$.

Example 8. Any subset of a partially ordered set under the restriction of $<$.

Definition 8. We say that $x \in X$ is *maximal* (respectively, *minimal*) with respect to a partial order $<$ if for all $y \in X$, $x \not< y$ (respectively, $y \not< x$).

Example 9. $X = \{$linearly independent subsets of $\mathbb{R}^n\}$ ordered by strict inclusion. Then the maximal elements of X are bases of $\mathbb{R}^n$ while the only minimal element is the empty set.

Definition 9. Let $(X, <)$ be a partially ordered set and let $A \subseteq X$. An *upper bound* (respectively *lower bound*) of A in X is an element $x \in X$ such that for all $a \in A$, $a < x$ or $a = x$ (resp. $x < a$ or $x = a$).

Definition 10. A *simple* or *linear* order is a partial order satisfying the additional condition

iii) $\forall x, y \in X$ $x < y$ or $y < x$ or $x = y$ (exclusively).

Example 10. $(\mathbb{R}, <)$ and all of its subsets.

Example 11 (Dictionary or Lexicographic ordering)**.** If $(X, <_X)$ and $(Y, <_Y)$ linearly ordered, then $X \times Y$ acquires a linear order by

$$(x_1, y_1) < (x_2, y_2) \qquad \text{if } x_1 <_X x_2 \text{ or } (x_1 = x_2 \text{ and } y_1 <_Y y_2)$$

Definition 11. A linear order $(X, <)$ is called **well-ordered** if every non-empty subset of X contains a minimal element.

Example 12. $\mathbb{Z}_{\geq 0} = 0, 1, 2, 3, ...$ is well-ordered.

Example 13. $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $[0, 1]$ are not well-ordered.

Example 14. $0, 1, 2, 3, ..., \infty, \infty + 1, ..., 2\infty, 2\infty + 1, ...$ is well-ordered (this is $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ in the dictionary ordering).

For $x \in X$, define the section $S_{<x} := \{y \in X | y < x\}$.

Theorem 2.5 (**Principle of Induction:**). *Let $(X, <)$ be a well-ordered set. Suppose that $X_0 \subseteq X$ is a subset satifying*

$$S_{<x} \subseteq X_0 \Rightarrow x \in X_0.$$

Then $X_0 = X$.

Proof. Let $Y := X \setminus X_0$. If Y is non-empty then Y has a minimal element $y \in Y$. But then $S_{<y} \subseteq X_0 \Rightarrow y \in X_0$ which is a contradiction. Thus Y is empty and $X_0 = X$. □

The principal of induction is of course most familiar for $X = \mathbb{Z}_{\geq 0}$. We mention in passing that recursion also makes sense for well-ordered sets (think Fibonacci sequences). This is a powerful result, because of the following.

Well-ordering Principle (Zermelo) Every set can be well-ordered.

The proof of this theorem is long, but not hard. It uses and is equivalent to,

The Axiom of Choice: Let $\{A_i\}_{i \in I}$ be a collection of disjoint, nonempty sets. Then there exists a set B containing exactly one element from each A_i.

The axiom of choice is a part of the most commonly accepted axiomatic foundation of set theory known as ZFC after Zermelo, Fraenkel and choice. The Axiom of Choice seems innocuous, but leads to some bizarre consequences.

Both of these are equivalent to the following

Zorn's Lemma: Let $(X, <)$ be partially ordered. Suppose that every linearly ordered subset of X has an upper bound in X. Then X contains a maximal element.

Theorem 2.6. *Every vector space V has a basis.*

Proof. Let X be the set of all linearly independent subsets of V, ordered by inclusion. A basis of V is the same as a maximal element in V. By Zorn's Lemma, it is enough to show:

Lemma 2.7. *Suppose that $\{A_i\}_{i \in I} \subseteq X$ is linearly ordered by inclusion. Then $A := \bigcup_{i \in I} A_i \in X$.*

Proof. Suppose that $v_1, ..., v_n \in A$. Then each $v_k \in A_{i_k}$ for some i_k. Since $\{A_{i_1}, ..., A_{i_n}\}$ is finite and linearly ordered, one of the A_{i_k} must be maximal. Thus $v_1, ..., v_n \in A_{i_k}$ must be linearly independent, and we conclude that $A \in X$. □

□

2.2 Equivalence Relations

Definition 12. An *equivalence relation* on a set X is a relation $\sim$ satisfying, for all $x, y \in X$

i) $x \sim x$ (reflexivity)

ii) $x \sim y$ implies $y \sim x$ (symmetry)

iii) $x \sim y$ and $y \sim z$ implies $x \sim z$ (transitivity)

Given $x \in X$, the *equivalence class* of x is

$$[x] := \{y \in X | x \sim y\}$$

Notice that $[x] = [y]$ if and only if $x \sim y$. The equivalence classes determine a partition of X into disjoint sets (prove this).

Let $E := \{[x] | x \in X\}$ be the set of equivalence classes. There is a canonical map

$$Q : X \to E, \quad x \mapsto [x]$$

called the *quotient map*.

Given any relation R on X, it is possible to generate an equivalence relation $\sim_R$ by the rule $x \sim_R y$ if and only if there exists a finite sequence $\{x_i \in X\}_{i=0}^n$ for $n \geq 0$ satisfying

(i) $x_0 = x$,
(ii) $x_n = y$ and,
(iii) $x_i R x_{i-1}$ or $x_{i-1} R x_i$ for all $i = 1, ..., n$.

Exercise 3. Prove that $\sim_R$ is an equivalence relation.

2.3 Cartesian Products and Coproducts

Given a collection of sets $\{X_\alpha\}_{\alpha \in I}$, one may form the *product set* or *Cartesian product*:

$$\prod_{\alpha \in I} X_\alpha = \{(x_\alpha) | x_\alpha \in X_\alpha \text{ for all } \alpha \in I\}$$

whose elements are I-tuples (x_α).

This is a pretty clear concept when I is finite, but what about when I is infinite? or uncountable? We can clarify this concept using something called a *universal property*.

For every $\beta \in I$, define the *projection map*

$$\pi_\beta : \prod_{\alpha \in I} X_\alpha \to X_\beta, \qquad \pi_\beta((x_\alpha)) = x_\beta$$

The product set satisfies the following universal property.

Proposition 2.8. *Suppose Y is a set and let $\{f_\alpha : Y \to X_\alpha\}_{\alpha \in I}$ be a collection of maps. Then there exists a unique map*

$$F : Y \to \prod_{\alpha \in I} X_\alpha$$

satisfying $f_\alpha = \pi_\alpha \circ F$. This property characterizes $\prod_{\alpha \in I} X_\alpha$ up to uniquely determined isomorphism.

Proof. The map $F : Y \to \prod_{\alpha \in I} X_\alpha$ is simply the coordinate map

$$F(y) = (f_\alpha(y)),$$

and it is pretty clear that $\pi_\alpha(F(y)) = f_\alpha(y)$ and that this determines F uniquely.

Now suppose that Z is a set equipped with maps $\tilde{\pi}_\alpha : Z \to X_\alpha$ possessing the universal property. Then there exist uniquely determined maps $F : Z \to \prod_{\alpha \in I} X_\alpha$ and $\tilde{F} : \prod_{\alpha \in I} X_\alpha \to Z$ such that

$$\rho_\alpha = \pi_\alpha \circ F$$

and

$$\pi_\alpha = \rho_\alpha \circ \tilde{F}$$

for every $\alpha \in I$. Composing, we have $\pi_\alpha = \pi_\alpha \circ F \circ \tilde{F}$ so by the uniqueness property

$$F \circ \tilde{F} = Id_{(\prod X_\alpha)}.$$

Similarly $\rho_\alpha = \rho_\alpha \circ \tilde{F} \circ F$ so

$$\tilde{F} \circ F = Id_Z.$$

□

We interpret can interpret Proposition (2.8) as providing a definition of the product set that is more concrete than "the set of I-tuples". This kind of argument, establishing that a universal property determines an object up to unique isomorphism, is called *proof by abstract nonsense* (this expression is often attributed to Saunders MacLane).

We note that the universal property possessed by the product is categorical in nature (speaks only of equalities and compositions of objects and morphisms) so it translates to define products of topological spaces, groups, modules, etc..

Following this philosophy, we define coproduct set.

Definition 13. Given a collection of sets $\{X_\alpha\}_{\alpha \in I}$, the *coproduct* (or disjoint union) is the set

$$\coprod_{\alpha \in I} X_\alpha = \{(x, \alpha) |\ \alpha \in I, \text{ and } x \in X_\alpha\}.$$

The maps $i_\alpha : X_\alpha \hookrightarrow \coprod_{\alpha \in I} X_\alpha$ defined by $i_\alpha(x) = (x, \alpha)$ are called the inclusion maps.

The coproduct satisfies the following universal property.

Proposition 2.9. *Suppose that Y is a set and let $\{g_\alpha : X_\alpha \to Y\}_{\alpha \in I}$ be a collection of maps. Then there is a unique map*

$$G : \coprod_{\alpha \in I} X_\alpha \to Y$$

such that for all $\alpha \in I$, $g_\alpha = G \circ i_\alpha$. This property characterizes the coproduct up to uniquely determined isomorphism.

Proof. Exercise. □

3 Topological Spaces

A *topological space* or simply a *space* consists of a set X and a collection τ of subsets of X, called the *open sets*, such that

1. $\emptyset$ and X are open,
2. Any union of open sets is open,
3. Any finite intersection of open sets is open.

It is conventional to denote a topological space (X, τ) simply by X when there is unlikely to be confusion about the topological structure (this will not be the case today).

The following topologies exist on any set X.

Example 15. *Trivial topology:* $\tau = \{\emptyset, X\}$.

Example 16. *Discrete topology:* $\tau = \wp(X)$ (the power set).

Example 17. *Finite complement topology:* $U \in \tau \Leftrightarrow X \setminus U$ is a finite set or $U = \emptyset$.

Definition 14. Suppose that τ and τ' are two topologies on a set X. If $\tau \subseteq \tau'$ then we say τ is *coarser* than τ', and τ' is *finer* than τ.

These relations determine a partial order on the set of all topologies on an underlying set X.

Example 18. The coarsest topology on X is the trivial topology. The finest topology on X is the discrete topology.

For most topologies, it is inconvenient to define the topology by specifying each and every open set. Instead, we usually "generate" the topology from a special collection of open sets.

Definition 15. A *subbasis (for a topology)* on a set X is a collection of subsets $S \subseteq \wp(X)$, whose union is the entire set X.

Proposition 3.1. *Let S be a subbasis for a topology. Define $\tau \subseteq \wp(X)$ to be the collection of all unions of finite intersections of sets in S. Then τ is a topology, called* ***the topology generated by S****.*

Proof. We must verify that τ satisfies the axioms.

1) The intersection of the empty collection is empty and by definition the union of all elements of S is X.
2) A union of unions of finite intersections is a union of finite intersections.
3) It is enough to prove that the intersection of any two elements $U, V \in \tau$ lies in τ. By definition, $U := \bigcup_{i\in I} U_i$ and $V := \bigcup_{j\in J} V_j$ where U_i and V_j are finite intersections of sets in S for all $i \in I$ and $j \in J$. Then

$$U \cap V = (\bigcup_{i\in I} U_i) \cap (\bigcup_{j\in J} V_j) = \bigcup_{I\times J}(U_i \cap V_j)$$

is a union of finite intersections, hence is an element of τ. □

Remark 1. Observe that the topology generated by S is the coarsest topology on X for which every element of S is open.

The fact that any subbasis generates a topology is useful theoretically, but it can be difficult in practice to determine whether a given set is open or not by the above definition. Thus we often use this intermediate approach.

Definition 16. A *basis for a topology* on X is a collection β of subsets of X which is a subbasis (i.e. the union of all elements in β is X) and which satisfies the following condition: For any pair of sets $U_1, U_2 \in \beta$, and any point $p \in U_1 \cap U_2$, there exists a set $U_3 \in \beta$ such that $p \in U_3 \subseteq U_1 \cap U_2$.

Definition 17. The *topology generated by* β is the collection τ of subsets $U \subseteq X$ satisfying of the following equivalent conditions.

(i) U is a union of finite intersections of elements of β (same as for subbasis).

(ii) U is a union of elements of β.

(iii) For all $p \in U$ there exists $V \in \beta$ such that $p \in V \subseteq U$.

Exercise 4. Show that (i), (ii), (iii) are equivalent for β a basis.

The value in considering topologies generated by a basis, rather than an arbitrary subbasis is that now we can use any of conditions (i), (ii) and (iii) to check if a set is open.

Example 19. Let $X = \mathbb{R}^n$ (or indeed any metric space) and let $\beta := \{B_\epsilon(p) | p \in X, \epsilon > 0\}$. Then β is the basis for a topology, called the *metric topology* on X (we will prove this when we return to metric spaces later).

Example 20. Let $X = \mathbb{C}^n$. For $f \in \mathbb{C}[x_1, ..., x_n]$ a polynomial function, let $U_f := \{p \in \mathbb{C}^n | f(p) \neq 0\}$. Then the set $\beta := \{U_f | f \in \mathbb{C}[x_1, ..., x_n]\}$ is a basis. The topology it generates is called the *Zariski topology.*

Proof. For polynomial functions f, g we have $f(p)g(p) = 0$ if and only if $f(p) = 0$ or $g(p) = 0$. Thus $U_f \cap U_g = U_{fg}$ so β is actually closed under intersection. □

Remark 2. When $n = 1$, the Zariski topology on $\mathbb{C}$ coincides with the finite complement topology (Prove this).

3.1 The Subspace Topology

Let (X, τ) be a topological space, and $A \subseteq X$ a subset. Then A inherits a topological structure τ_A from (X, τ) defined by

$$\tau_A := \{U \cap A | U \in \tau\}.$$

We call τ_A the *subspace topology.*

Proposition 3.2. *The subspace topology τ_A is a topology on A.*

Proof. 1) $\emptyset = \emptyset \cap A$ and $A = X \cap A$ are elements of τ_A.

Now let $\{U_\alpha \cap A\}_{\alpha \in I}$ be an arbitrary collection of elements of τ_A, where U_α are open sets in X. Then

2) $\bigcup_{\alpha \in I}(U_\alpha \cap A) = (\bigcup_{\alpha \in I} U_\alpha) \cap A \in \tau_A$ because intersections distribute over unions.

3) $\bigcap_{\alpha \in I}(U_\alpha \cap A) = (\bigcap_{\alpha \in I} U_\alpha) \cap A \in \tau_A$ because intersections commute with intersections. □

Example 21. Any subspace of $\mathbb{R}^n$ inherits a Euclidean topology.

Example 22. Any subspace of $\mathbb{C}^n$ inherits a Zariski topology.

The subspace topology can also be defined using a universal property. Let $i : A \hookrightarrow X$ be the inclusion map of sets.

Proposition 3.3. *The subspace topology on A is the coarsest topology for which the inclusion map is continuous.*

Proof. Recall that a map is continuous if the pre-image of open sets is open. Let τ_A' be a topology on A and suppose that $i : (A, \tau_A') \hookrightarrow (X, \tau)$ is continuous. Then for any open set $U \subseteq X$, the preimage $i^{-1}(U) = U \cap A$ must be continuous. So by the definition of the subspace topology τ_A, we have $\tau_A \subseteq \tau_A'$. □

Definition 18. A subset $A \subset X$ is called a *discrete subset* if A the subspace topology equals the discrete topology.

3.2 Closed Sets and Limits

Definition 19. A subset $A \subseteq X$ of a topological space is called *closed* if the complement $A^c = X \setminus A$ is open.

Proposition 3.4. *Let X be a topological space. Then*

i) $\emptyset$ and X are closed.

ii) Arbitrary intersections of closed sets are closed.

iii) Finite unions of closed sets are closed.

Proof. i) $(\emptyset)^c = X$ and $X^c = \emptyset$ are open.

The remaining cases follow by de Morgan's laws

ii) If A_α is closed then

$$\Big(\bigcap_{\alpha \in I} A_\alpha\Big)^c = \bigcup_{\alpha \in I} (A_\alpha)^c$$

is open.

iii) If A_i is closed then

$$\Big(\bigcup_{i=1}^{n} A_i\Big)^c = \bigcap_{i=1}^{n} (A_i)^c$$

is open. □

Of course, if one knows the closed sets of a topological space, one can recover the open sets by taking complements. Thus one may define a topology on X to be a class of closed sets satisfying properties i), ii) and iii) of Proposition 3.4, and this point of view is sometimes more convenient.

For $A \subset X$ a subset of a topological space, the *closure of* A is the subset of X

$$\bar{A} := \bigcap_{A \subseteq C,\, C \text{ closed}} C$$

Observe that $\bar{A}$ is closed because it is the intersection of closed sets. Thus we may also describe $\bar{A}$ as the smallest closed set containing A.

Similarly, one defines the *interior of* A to be the largest open set contained in A.

$$int(A) := \bigcup_{U \subseteq A,\, U \text{ open}} U = (\overline{A^c})^c$$

Example 23. Let $(a, b] \subset \mathbb{R}$ with the Euclidean topology. Then $int((a, b]) = (a, b)$ and $\overline{(a, b]} = [a, b]$.

An *(open) neighbourhood* of a point $p \in X$ is an open set U containing p.

Definition 20. Let A be a subset of a topological space X. An element $p \in X$ is called a *limit point of* A if for all open nbhds $p \in U$ we have

$$(U \setminus \{p\}) \cap A \neq \emptyset$$

Example 24. If $A = (a, b] \subset \mathbb{R}$, then the set of limit points is $[a, b]$

Example 25. If $A = \{1/n | n = 1, 2, 3, ...\} \subset \mathbb{R}$, then the only limit point is 0.

Theorem 3.5. *Let A be a subset of a topological space X and let A' be the set of limit points of A. Then*

$$A' \cup A = \bar{A}$$

Corollary 3.6. *A is closed if and only if it contains all of its limit points.*

Proof of Theorem. Suppose $p \notin \bar{A}$. Then $(\bar{A})^c$ is a nbhd of p and $(\bar{A})^c \cap A = \emptyset$ so $p \notin A'$ and consequently $p \notin A \cup A'$.

Now suppose that $p \notin A \cup A'$. Then there exists a nbhd U of p with $(U \setminus \{p\}) \cap A = \emptyset$. Thus $U \cap A = \emptyset$ so $U^c \supseteq A$. Since U^c is closed we have $U^c \supseteq \bar{A}$ so $p \notin \bar{A}$. □

3.3 Convergence and the Hausdorff Condition

A sequence $(x_n)_{n=1}^{\infty}$ of points in a topological space X is said to *converge to a limit* $L \in X$ if for every open nbhd U of L the *tail of the sequence* lies in U. That is, for every U there exists an integer N such the $x_n \in U$ for $n > N$.

It is possible for a sequence to converge to more than one point. For example, in the trivial topology every sequence converges to every point!

Definition 21. A topological space X is called Hausdorff if for any two distinct points $p_1, p_2 \in X$, there exists open nbhds U_1, U_2 of p_1, p_2 respectively, such that $U_1 \cap U_2 = \emptyset$.

We say that a Hausdorff topology *separates* points.

Proposition 3.7. *A sequence $(x_n)_{n=1}^{\infty}$ in a Hausdorff space can converge to at most one point.*

Proof. Let $(x_n)_{n=1}^{\infty}$ converge to L, and choose $L' \neq L$. Then there exist disjoint open nbhds U and U' of L and L' respectively. Since the tail of the sequence lies in U it cannot lie in U'. □

Example 26. The Euclidean topology on $\mathbb{R}^n$ is Hausdorff, because $p_1, p_2 \in \mathbb{R}^n$ with $d(p_1, p_2) = \epsilon > 0$, then $B_{\epsilon/3}(p_1)$ and $B_{\epsilon/3}(p_2)$ are disjoint open nbhds.

Example 27. The Zariski topology on $\mathbb{C}^n$ is *not* Hausdorff. In fact any two non-empty open sets intersect, because $U_{f_1} \cap U_{f_2} = U_{f_1 f_2}$ is empty if and only if $f_1 f_2 = 0$ if and only if $f_1 = 0$ or $f_2 = 0$.

Proposition 3.8. *Subspaces of Hausdorff spaces are Hausdorff.*

Proof. Let $A \subset X$ and X is Hausdorff. If $p_1, p_2 \in A$ are distinct points, then there exist open nbhds $U_1, U_2 \subset X$ which separate p_1 and p_2. Then $U_1 \cap A$ and $U_2 \cap A$ are open nbhds in A separating p_1 and p_2. □

Proposition 3.9. *Finite subsets of Hausdorff spaces are closed.*

Proof. It is enough to prove that one element sets are closed (because finite unions of closed sets are closed). Let $p \in X$. For every $q \in X$ distinct from p, there exists an open set U_q disjoint from a nbhd of p, so $p \notin U_q$. Thus

$$X \setminus \{p\} = \bigcup_{q \in X \setminus \{p\}} U_q$$

is open, so $\{p\}$ is closed. □

3.4 Continuity

Definition 22. Let X, Y be topological spaces. A map $f : X \to Y$ is *continuous* if for all open sets $U \subseteq Y$ the pre-image $f^{-1}(U)$ is open in X. Equivalently, f is continuous if for all closed sets $C \subset Y$ the pre-image $f^{-1}(C)$ is closed.

Proposition 3.10. *If the topology of Y is generated by a subbasis S then $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is open for all $B \in S$.*

Proof. A general open set in Y is a union of finite intersections of sets in S. That is of the form

$$U = \bigcup_{\alpha \in I} \Big(\bigcap_{i=1}^{n_\alpha} B_{\alpha,i} \Big)$$

where each $B_{\alpha,i} \in S$. Taking the preimage commutes with unions and intersections so:

$$f^{-1}(U) = f^{-1}\Big(\bigcup_{\alpha \in I} \big(\bigcap_{i=1}^{n_\alpha} B_{\alpha,i} \big) \Big) = \bigcup_{\alpha \in I} \Big(\bigcap_{i=1}^{n_\alpha} f^{-1}(B_{\alpha,i}) \Big)$$

which is open if $f^{-1}(B_{\alpha,i})$ is open. □

Exercise 5. Show that a map $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous in the Euclidean topology iff it is continuous in the $\epsilon - \delta$ sense.

Definition 23. A continuous map $f : X \to Y$ is called a *homeomorphism*[1] if it is bijective and the inverse map $f^{-1} : Y \to X$ is also continuous. In this case we say X and Y are *homeomorphic* (this is the notion of isomorphism in the category of topological spaces).

A homeomorphism induces a one-to-one correspondence between open sets, because if $V \subset X$ is open, then $f(V) = (f^{-1})^{-1}(V)$ is open in Y.

[1]The word homeomorphism comes from the Greek words homoios = similar and morph = shape or form.

Example 28. Let $f : (-1, 1) \to \mathbb{R}$ sending $f(x) = tan(\pi x/2)$. Then f is a homeomorphism, with inverse $f^{-1}(y) = (2/\pi)tan^{-1}(y)$..

Theorem 3.11. *Let X, Y, Z be topological spaces.*

a) ***Constant maps:*** *are continuous.*

b) ***Inclusion maps:*** *If $A \subseteq X$ is a subspace, then the inclusion map is continuous*

c) ***Composition:*** *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f$ is continuous.*

d) ***Restriction of domain:*** *If $A \subseteq X$ is a subspace, and $f : X \to Y$ is continuous, then the restriction $f|_A : A \to Y$ is continuous.*

e) ***Restriction of codomain:*** *If $B \subseteq Y$ is a subspace and $f : X \to Y$ is continuous with $f(X) \subseteq B$ then the restricted map $g : X \to B$ defined by $f(x) = g(x)$ is continuous.*

Proof. a) If $f : X \to Y$ is constant, and $U \subset Y$ then $f^{-1}(U) = \emptyset$ or X, so the preimage of *any* set is open.

b) Done previously.

c) If $U \subset Z$ is open, then $g^{-1}(U) \subset Y$ is open, then $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \subset X$ is open.

d) This follows from b) and c), because $f|_A$ is the composition of f with the inclusion map $i : A \hookrightarrow X$.

e) The general open set in B is of the form $U \cap B$ for U open in Y, and $g^{-1}(U \cap B) = f^{-1}(U \cap B) = f^{-1}(U)$ is open. □

3.5 Patching Continuous Functions Together

Lemma 3.12. *Let X be a topological space, $B \subseteq A \subseteq X$ a chain of subspaces.*

(i) If A is open in X, then B is open in A iff B is open in X.
(ii) If A is closed in X, then B is closed in A iff B is closed in X.

To see why this condition on A is necessary, consider the case $X = \mathbb{R}$ with Euclidean topology, $A = [0, 1]$, $A' = (1/2, 3, 2)$ and $B = A \cap A' = (1/2, 1]$. Then B is open as a subset of A, is closed as a subset of A' but is neither open nor closed as a subset of $\mathbb{R}$.

Proof. We prove (i) only, because (ii) is similar. Certainly if B is open in X, then $B \cap A = B$ is open in A by definition of the subspace topology.

Now suppose A is open in X. If B is open in A then by the definition of the subspace topology, $B = U \cap A$ for some U open in X. But then B is an intersection of open sets in X, hence open in X. □

Theorem 3.13. *Let X be a topological space and let $\{U_\alpha\}_{\alpha\in I}$ be an* open cover *(i.e. the U_α are open sets and their union is X). A map $f : X \to Y$ is continuous if and only if its restrictions $f_\alpha := f|_{U_\alpha} : U_\alpha \to Y$ are continuous for all $\alpha \in I$.*

Proof. We proved already that if f is continuous, then its restrictions f_α are continuous.

Now assume the f_α are all continuous. Let $V \subseteq Y$ be an open set. Then

$$\begin{aligned} f^{-1}(V) &= f^{-1}(V) \cap (\bigcup_{\alpha\in I} U_\alpha) \\ &= \bigcup_{\alpha\in I} (f^{-1}(V) \cap U_\alpha) \\ &= \bigcup_{\alpha\in I} f_\alpha^{-1}(V). \end{aligned}$$

The $f_\alpha^{-1}(V)$ are open in U_α hence open in X by Lemma 3.12. Thus $f^{-1}(V)$ is a union of open sets, hence open. □

Theorem 3.13 allows us to define continuous maps *locally*. That is, if we have a collection of continuous maps f_α defined on each U_α which agree on overlaps, then we can patch them together to define a map on X. A similar results is valid for closed covers:

Exercise 6. Let X be a topological space and $\{C_i\}_{i=1}^n$ be a finite *closed cover*. Show that $f : X \to Y$ is continuous if and only if $f|_{C_i}$ is continuous for all i.

3.6 The Initial Topology

Definition 24. Let X be a set and let $\{f_\alpha : X \to Y_\alpha\}_{\alpha\in I}$ a collection of maps of sets, where the Y_α are topological spaces. The *initial topology* on X is the coarsest topology for which the f_α are all continuous.

Explicitly, the initial topology on X is the topology generated by the subbasis

$$\mathcal{S} := \{f_\alpha^{-1}(U_\alpha) \mid U_\alpha \subseteq Y_\alpha \text{ is open}\}$$

Example 29. The subspace topology of $A \subseteq X$ is the initial topology with respect to inclusion $i : A \hookrightarrow X$.

Example 30. Let $X = \mathcal{H}$ (infinite dimensional, separable) Hilbert space, and let $\{L : \mathcal{H} \to \mathbb{C}\}$ be the set of bounded linear functionals. The initial topology is called the weak topology on $\mathcal{H}$ (this is coarser than the norm (metric) topology).

Example 31. Let $X = \mathbb{C}^n$ and $\{f : \mathbb{C}^n \to \mathbb{C}\}$ be the set of polynomial functions, and give $\mathbb{C}$ the finite complement topology. Then the initial topology with respect to these maps is the Zariski topology.

3.7 Product Topology

Definition 25. Let $\{X_\alpha\}$ be a collection of topological spaces. The *product space*, denoted

$$\prod_{\alpha \in I} X_\alpha$$

is the topological space with underlying set the cartesian product set, equipped with the initial topology with respect to the projection maps

$$\pi_{\alpha_0} : \prod_{\alpha \in I} X_\alpha \to X_{\alpha_0}.$$

This topology is generated by a *basis* of open sets of the form:

$$\prod_{\alpha \in I} U_\alpha \subseteq \prod_{\alpha \in I} X_\alpha$$

where U_α is open in X_α and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in I$ (prove this).

The product topology satisfies (and is determined up to unique isomorphism by) the following universal property:

Proposition 3.14. *Let Z be a topological space and let $\{f_\alpha : Z \to X_\alpha\}_{\alpha \in I}$ be a collection of continuous maps. Then there exists a unique continuous map*

$$F : Z \to \prod_{\alpha \in I} X_\alpha$$

such that $f_\alpha = \pi_\alpha \circ F$. The f_α are called the coordinate functions of F.

Proof. Homework Exercise. □

Example 32. In the Euclidean topologies, $\mathbb{R}^n \cong \prod_{i=1}^n \mathbb{R}$ (prove this).

Example 33. In the Zariski topology, $\mathbb{C}^n \ncong \prod_{i=1}^n \mathbb{C}$ (prove this).

Example 34. The product topology on $\mathbb{R}^\infty = \prod_{i=1}^\infty \mathbb{R}$ contains (and is generated by) open sets of the form $(a_1, b_1) \times ... \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times ...$ where n is an arbitrary positive integer. Warning: the subset $\prod_{i=1}^\infty (0, 1)$ is *not* an open set in the product topology.

Proposition 3.15. *Let X and Y be topological spaces and let $p \in Y$ be a point. Then the subspace $X \times \{p\} \subseteq X \times Y$ is homeomorphic to X.*

Proof. Consider the map $F : X \to X \times \{p\}$ sending $F(x) = (x, p)$. This clearly a bijection. To see that F is a homeomorphism, observe that the coordinate functions Id_K and the constant map p are both continuous, so F is continuous as a map into $X \times Y$ are remains continuous upon restricting the codomain to $X \times \{p\}$.

Conversely, the inverse map F^{-1} is the composition of the inclusion map $X \times \{p\} \hookrightarrow X \times Y$ and the projection map $X \times Y \to X$, so F^{-1} is a composition of continuous maps, hence continuous. □

3.8 The Final Topology

Let X be set and let $\{f_\alpha : Y_\alpha \to X\}_{\alpha \in I}$ be a collection of maps of sets, where the Y_α are topological spaces. The *final topology* on X is the finest topology on X such that f_α is continuous for every α.

Proposition 3.16. *The open sets in the final topology are equal to the collection $\tau := \{V \in X | f^{-1}(V)$ is open in Y_α for all $\alpha \in I\}$.*

Proof. It is clear that if τ is a topology, it must be the finest topology for which the f_α are all continuous. Thus it suffices to check that τ satisfies the axioms of a topology.

For any $\alpha \in I$:

- $f_\alpha^{-1}(\emptyset) = \emptyset$ and $f^{-1}(X) = Y_\alpha$ are both open in Y_α, so $\emptyset$ and X lie in τ.
- If $\{V_j\}_{j \in J} \subseteq \tau$, then $f_\alpha^{-1}(\bigcup_{j \in J} V_j) = \bigcup_{j \in J} f_\alpha^{-1}(V_j)$ is open in Y_α, so $\bigcup_{j \in J} V_j \in \tau$.
- Finite intersections is similar to unions.

□

Exercise 7. Show that $C \subset X$ is closed in the final topology exactly when $f_\alpha^{-1}(C)$ is closed for all $\alpha \in I$.

3.9 Quotient Topology

Let $\sim$ be an equivalence relation on a topological space X and denote by $X/\sim$ the set of equivalence classes (I denoted this by E before), and let

$$Q : X \to X/\sim$$

be the quotient map $Q(x) = [x]$.

Definition 26. The *quotient topology* on $X/\sim$ is the final topology on $X/\sim$ with respect to the map Q. In particular, $U \subseteq X/\sim$ is open if and only if $Q^{-1}(U)$ is open. We call $X/\sim$ the *quotient space.*

Example 35. $X = [0,1]$ with Euclidean topology and $\sim$ is the equivalence relation generated by $0 \sim 1$. Then the quotient space

is homeomorphic to the circle $S^1 := \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$.

Example 36. A torus. Let $X := [0,1] \times [0,1]$ (with Euclidean topology). Let $\sim$ be the equivalence relation generated by $(x,1) \sim (x,0)$ and $(0,y) \sim (1,y)$ for all $x, y \in [0,1]$. Then the quotient space is homeomorphic to the torus $S^1 \times S^1$.

Example 37. $X = S^2$ is the unit sphere in $\mathbb{R}^3$ with the Euclidean geometry, which with think of as the surface of the globe. Choose an equivalence relation $x \sim y$ if they have the same latitude. Then the quotient space $X/\sim$ is homeomorphic to an interval (say $[-90, 90]$).

By some abstract nonsense, the following universal property characterizes the quotient space up to unique isomorphism.

Proposition 3.17. *Let X is a topological space with equivalence relation $\sim$ and quotient space $Q : X \to X/\sim$. Let $f : X \to Y$ be a continuous map with the property that for $a, b \in X$,*

$$a \sim b \Rightarrow f(a) = f(b).$$

Then there exists a unique continuous map $g : X/\sim \to Y$ for which the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle Q}\searrow \quad \nearrow{\scriptstyle g} & \\ & X/\sim & \end{array} \tag{1}$$

commutes. That is to say, $f = g \circ Q$. We say that f ***descends*** *to g.*

Proof. It is clear at the level of sets that commutativity of the diagram forces $g([x]) = f(x)$. Since

$$[x] = [y] \Leftrightarrow x \sim y \Rightarrow f(x) = f(y) \Rightarrow g([x]) = g([y]),$$

we see that g is well defined as a map of sets.

To prove that g is continuous, for any $U \subset Y$ open we must show $g^{-1}(U)$ is open. We know that

$$Q^{-1}(g^{-1}(U)) = (g \circ Q)^{-1}(U) = f^{-1}(U)$$

which is open because f is continuous. It follows from the definition of the quotient topology that $g^{-1}(U)$ is open. □

3.10 Coproducts

Given a collection $\{X_\alpha\}_{\alpha \in I}$ of topological spaces, the *coproduct space*, denoted $\coprod_{\alpha \in I} X_\alpha$ is constructed as follows. The set underlying $\coprod_{\alpha \in I} X_\alpha$ is

$$\{(x, \alpha) | \alpha \in I \text{ and } x \in X_\alpha\}$$

the "disjoint union" of the X_α. The topology is the final topology with respect to the inclusion maps i_α,

$$i_{\alpha_0} : X_{\alpha_0} \hookrightarrow \coprod_{\alpha \in I} X_\alpha, \qquad i_{\alpha_0}(x) = (x, \alpha_0).$$

Informally, we think of the X_α as disjoint subspaces of the coproduct with inclusion map i_α and declare a subset $U \subseteq \coprod_{\alpha \in I} X_\alpha$ to be open if $U \cap X_\alpha$ is open in X_α.

Example 38. Let $[0, 1]$ be the unit interval with Euclidean topology. Then $[0, 1] \coprod [0, 1]$ is homeomorphic to a disjoint union of closed intervals, say $[0, 1] \cup [2, 3]$.

The coproduct is determined, up to unique isomorphism, by the following universal property.

Proposition 3.18. *The coproduct satisfies the following universal property. Given a topological space Y and a collection of continuous maps $\{f_\alpha : X_\alpha \to Y\}_{\alpha \in I}$, there exists a unique continuous map $F : \coprod_{\alpha \in I} X_\alpha \to Y$ such that the following diagram commutes for every $\alpha_0 \in I$:*

$$\begin{array}{ccc} X_{\alpha_0} & \xrightarrow{f_{\alpha_0}} & Y \\ & {\scriptstyle i_{\alpha_0}}\searrow \quad \nearrow{\scriptstyle F} & \\ & \coprod_{\alpha \in I} X_\alpha & \end{array} \tag{2}$$

Proof. Commutativity of the diagram requires that $F((x, \alpha)) = f_\alpha(x)$ and this is clearly well-defined as a map of sets. To see that F is continuous, choose $U \subset Y$ open. Then for all α

$$i_\alpha^{-1}(F^{-1}(U)) = f_\alpha^{-1}(U)$$

is open, so by definition of the final topology, $F^{-1}(U)$ is open and we conclude that F is continuous. $\square$

Remark 3. The proofs of Propositions 3.17 and 3.18 are very similar. In fact, both are special cases of a more general construction: that of a colimit. We will not explore this more sophisticated notion. Every colimit can be produced by first taking a coproduct, and then taking a quotient, and we will follow this approach.

Example 39. Consider the complex plane $\mathbb{C}$ (with either the Euclidean or Zariski topology). Consider the coproduct $\mathbb{C} \coprod \mathbb{C}$ and use notation $z \in \mathbb{C}$ for an element in the first copy of $\mathbb{C}$ and $w \in \mathbb{C}$ for an element the second copy. Let $\sim$ be the equivalence relation generated by

$$z \sim 1/w \text{ for all } z \neq 0.$$

If $\mathbb{C} \cong \mathbb{R}^2$ is given the Euclidean topology, the quotient $(\mathbb{C} \coprod \mathbb{C})/\sim$ is homeomorphic to S^2 in the Euclidean topology and is called the *Riemann sphere*. If $\mathbb{C}$ carries the Zariski topology, $(\mathbb{C} \coprod \mathbb{C})/\sim$ carries the finite-complement topology, and is called the *complex projective line.*

Example 40. The space $D^n := \{\vec{x} \in \mathbb{R}^n | |\vec{x}| \leq 1\}$ is called the n-disk or n-cell (we adopt the convention that D^0 is a point). It contains the $(n-1)$-sphere $S^{n-1} := \{\vec{x} \in \mathbb{R}^n | |\vec{x}| = 1\}$ as a subspace. Let X be a topological space and let $f : S^{n-1} \to X$ be a continuous map. We form a new space

$$\hat{X} := (X \coprod D^n)/\sim$$

where $\sim$ is the equivalence relation generated by $\vec{x} \sim f(\vec{x})$, for all $\vec{x} \in S^{n-1} \subset D^n$. We say that $\hat{X}$ is obtained from X by *gluing on an n-cell.*

Observe that as a set, $\hat{X}$ is the dijoint union of X with the interior of D^n, but the topology is not the coproduct of these subspaces.

Definition 27. We define n-dimensional *cell-complexes* (or *CW-complexes*) inductively.

A 0-dimensional cell-complex is a set with the discrete topology, X_0 (i.e. a coproduct of 0-cells).

An n-dimensional cell-complex X_n is a space constructed by gluing n-cells to a $(n-1)$-dimensional cell-complex using maps $f : S^{n-1} \to X_{n-1}$.

Example 41. S^n is a n-dimensional cell-complex. Simply glue D^n to D^0 by the constant map $S^{n-1} \to D^0$.

Example 42. D^n is an n-dimensional cell-complex. Glue D^n to S^{n-1} by the identity map $S^{n-1} \to S^{n-1}$.

Example 43. The torus is a 2-dimensional cell-complex. Recall the definition of the torus as a quotient. This gives the same space as the cell-complex with one 0-cell and two 1-cells and the one 2-cell. The 2-cell is glued according to the map $S^1 \to (S^1 \coprod S^1)/\sim$ which winds around the first loop, then the second loop, then backwards around the first and backwards around the second.

4 Metric Topologies

Definition 28. A *metric* on a set X is a function

$$d : X \times X \to \mathbb{R}$$

satisfying: $\forall a, b, c \in X$

- $d(a, b) \geq 0$ and $d(a, b) = 0 \Leftrightarrow a = b$.
- $d(a, b) = d(b, a)$.
- $d(a, b) + d(b, c) \geq d(a, c)$.

The function d is called the *distance* or *metric* function.

The third condition above is called the *triangle inequality.* This terminology comes from the following helpful diagram:

Definition 29. The *metric topology* on a metric space (X, d) is generated by the subbasis,

$$\mathcal{B} := \{B_\epsilon(a) | a \in X, \epsilon > 0\}$$

where $B_\epsilon(a) = \{x \in X | d(x, a) < \epsilon\}$ is the epsilon-ball centered at a.

Proposition 4.1. *The subbasis $\mathcal{B}$ above is actually a basis. This means in particular that $U \subset X$ is open in the metric topology if for every $a \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(a) \subset U$.*

Proof. Let $B_{r_1}(a)$ and $B_{r_2}(b)$ be two basic open sets, and suppose $c \in B_{r_1}(a) \cap B_{r_2}(b)$. We must show that there exists a basic open set containing c and contained in $B_{r_1}(a) \cap B_{r_2}(b)$.

By definition then, $d(a, c) < r_1$ and $d(b, c) < r_2$. It follows that

$$r := min(r_1 - d(a, c), r_2 - d(b, c)) > 0.$$

Claim: $B_r(c) \subseteq B_{r_1}(a) \cap B_{r_2}(b)$.

If $p \in B_r(c)$, then

$$d(a, p) \leq d(a, c) + d(c, p) < d(a, c) + r \leq r_1,$$

so $p \in B_{r_1}(a)$. Similarly,

$$d(b, p) \leq d(b, c) + d(c, p) < d(a, c) + r \leq r_2,$$

so $p \in B_{r_2}(b)$. $\square$

Definition 30. A topological space (X, τ) is called *metrizable* if there exists a metric on X whose metric topology agrees with τ.

When is a topology metrizable? We will obtain a partial answer to this question later in the course with the *Urysohn Metrization Theorem.*

Proposition 4.2. *Let $A \subset X$ be a subset of a metric space (X, d). Then A is a metric space under the restriction $d|_A$ of d and the metric topology agrees with the subspace topology on A.*

Proof. The axioms of a metric are clearly preserved under restriction.

Also $B_\epsilon^{d|_A}(a) = B_\epsilon^d(a) \cap A$, so a basic open set in metric topology is open in the subspace topology, so the metric topology finer.

On the other hand, a general open set in the subspace topology has the form $U \cap A$ where U is open in X. Given $p \in A \cap U$, we may find $\epsilon > 0$ such that $B_\epsilon^d(p) \subset U$ and thus $B_\epsilon^{d|_A}(p) \subseteq A \cap U$, so $U \cap A$ is open in the metric topology, so the metric topology is coarser so the two topologies must coincide. □

Lemma 4.3. *Let d_1 and d_2 be two metrics on the same set X. Suppose that for every $a \in X$, $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$B_\epsilon^{d_1}(a) \supseteq B_\delta^{d_2}(a).$$

Then the metric topology of d_2 is finer than the metric topology of d_1 (in fact this is if and only if).

Proof. Let U be a open with respect to d_1. Then for all $a \in U$, there exists $\epsilon > 0$ such that $B_\epsilon^{d_1}(a) \subseteq U$. Thus there exists $\delta > 0$ such that $B_\delta^{d_2}(a) \subseteq U$, so U is also open with respect to d_2. □

Example 44. Let $X = \mathbb{R}^n$. The Euclidean metric

$$d_e(\vec{a}, \vec{b}) = \sqrt{(a_1 - b_1)^2 + ... + (a_n - b_n)^2}$$

and the taxi-cab metric

$$d_t(\vec{a}, \vec{b}) = \sum_{i=1}^{n} |a_i - b_i|$$

both determine the Euclidean topology on $\mathbb{R}^n$, because

$$d_e(\vec{a}, \vec{b})^2 = \sum_{i=1}^{n} |a_i - b_i|^2 \le (\sum_{i=1}^{n} |a_i - b_i|)^2 = d_t(\vec{a}, \vec{b})^2$$

While on the other hand,

$$d_t(\vec{a}, \vec{b}) = \sum_{i=1}^{n} |a_i - b_i| \le \sum_{i=1}^{n} d_e(\vec{a}, \vec{b}) = n d_e(\vec{a}, \vec{b})$$

So for any $\vec{a} \in \mathbb{R}^n$, and $\epsilon > 0$,

$$B_{\epsilon/n}^{d_e}(\vec{a}) \subseteq B_\epsilon^{d_t}(\vec{a}) \subseteq B_\epsilon^{d_e}(\vec{a})$$

The following proposition allows us to replace any metric by a bounded one - this is useful in constructions.

Proposition 4.4. *Let d be a metric on a set X, and define $\bar{d} : X \times X \to \mathbb{R}$ by*

$$\bar{d}(a,b) = \min\{d(a,b), 1\}.$$

Then $\bar{d}$ is a metric and it determines the same topology as d. We call $\bar{d}$ the standard bounded metric associated to d.

Proof. We begin by showing that $\bar{d}$ is a metric. Clearly $\bar{d}(a,b) \geq 0$, $\bar{d}(a,b) = 0 \Leftrightarrow a = b$ and $\bar{d}(a,b) = \bar{d}(b,a)$. It remains to verify the triangle inequality:

$$\bar{d}(a,b) + \bar{d}(b,c) \overset{?}{\geq} \bar{d}(a,c)$$

If $\bar{d}(a,b) = 1$ or $\bar{d}(b,c) = 1$ then the $LHS \geq 1 \geq RHS$ so we are okay. Otherwise, we have:

$$\bar{d}(a,b) + \bar{d}(b,c) = d(a,b) + d(b,c) \geq d(a,c) \geq \bar{d}(a,c)$$

completing the proof that $\bar{d}$ is a metric. To prove that $\bar{d}$ determines the same topology as d, first observe that for $\delta < 1$, $B^d_\delta(a) = B^{\bar{d}}_\delta(a)$. Thus for any $a \in X$ and any $\epsilon > 0$,

$$B^{\bar{d}}_\delta(a) \subseteq B^d_\epsilon(a) \subseteq B^{\bar{d}}_\epsilon(a)$$

where $\delta = min\{\epsilon, 1/2\}$. Applying Lemma 4.3 completes the proof. □

4.1 Mapping Spaces and Uniform Convergence

For a pair of sets I and X, denote by

$$X^I := Maps_{Sets}(I, X)$$

the *set* of maps of sets from I to X. There is an isomorphism of sets

$$X^I \cong \prod_{\alpha \in I} X, \qquad f \mapsto (f(\alpha))_{\alpha \in I},$$

so it is possible to think an element of X^I as either a function or as an I-tuple. We use notation

$$f(\alpha) = f_\alpha$$

to emphasize this correspondence.

Let d be a metric on X and let $\bar{d} = min\{d, 1\}$ be standard bounded metric associated to d.

Proposition 4.5. *The function $\rho : X^I \times X^I \to \mathbb{R}$ defined by*

$$\begin{aligned} \rho(f,g) &= sup\{\ \bar{d}(f_\alpha, g_\alpha) \mid \alpha \in I\} \\ &= sup\{\ 1, d(f(\alpha), g(\alpha)) \mid \alpha \in I\} \end{aligned}$$

*is a metric on X^I called the **uniform metric**.*

Proof. First observe that $\{\bar{d}(f_\alpha, g_\alpha)|\alpha \in I\}$ is bounded above by 1, so its supremum exists and ρ is well-defined (this is why we worked with the bounded metric).

(1) (Positivity) $\rho(f, g) \geq 0$ because it is the supremum of a nonnegative set and

$$\begin{aligned} \rho(f,g) = 0 &\Leftrightarrow d(f_\alpha, g_\alpha) = 0, \quad \forall \alpha \in I \\ &\Leftrightarrow f_\alpha = g_\alpha, \quad \forall \alpha \in I \\ &\Leftrightarrow f = g. \end{aligned}$$

(2) (Symmetry) $\rho(f, g) = sup\{\bar{d}(f_\alpha, g_\alpha)|\alpha \in I\} = sup\{\bar{d}(g_\alpha, f_\alpha)|\alpha \in I\} = \rho(g, f)$.

(3) (Triangle inequality) By the definition of the supremum, $\forall \epsilon > 0$, $\exists \alpha \in I$ such that

$$\begin{aligned} \rho(f,h) &\leq \bar{d}(f_\alpha, h_\alpha) + \epsilon \\ &\leq \bar{d}(f_\alpha, g_\alpha) + \bar{d}(g_\alpha, h_\alpha) + \epsilon \\ &\leq \rho(f,g) + \rho(g,h) + \epsilon \end{aligned}$$

and because this is true for all positive ϵ, we conclude

$$\rho(f,h) \leq \rho(f,g) + \rho(g,h)$$

□

Example 45. Consider $\mathbb{R}^I$ where $\mathbb{R}$ has Euclidean metric. If 0 represents the constant zero function, then $B^\rho_\epsilon(0)$ is the set of functions $f : I \to \mathbb{R}$ whose range lies in the interval $(-\epsilon, \epsilon)$.

Example 46. If Y is a topological space and (X, d) is a metric space, then set of continuous maps $Cont(Y, X)$ inherits a uniform metric by restriction.

Proposition 4.6. *The uniform metric topology (or simply* uniform topology*) on X^I is finer than the product topology under the bijection $X^I \cong \prod_{\alpha \in I} X$. If I is finite, then the uniform and product topologies agree.*

We first prove a lemma.

Lemma 4.7. *Let τ and τ' be two topologies on the same set X. Then τ' is coarser than τ if and only if for every set $U \in \tau$ and every point $p \in U$, there exists $V_p \in \tau'$ such that $p \in V_p \subseteq U$.*

Proof. If τ' is finer than τ, then $U \in \tau$ implies $U \in \tau'$ so may choose $V_p = U$.

Conversely, for $U \in \tau$, we have

$$U = \bigcup_{p \in U} V_p \in \tau'.$$

□

Proof of Proposition 4.6. To show that the uniform topology is finer than the product topology, it suffices to prove that every basic open set in the product topology is also open in the uniform topology.

Recall that a basic open set for the product topology is one of the form

$$U = \prod_{\alpha \in I} U_\alpha$$

where each U_α is open in $X_\alpha = X$ (equipped with the metric topology of $\bar{d}$) and such that $U_\alpha = X$ for all but finitely many $\alpha \in I$, say $\{\alpha_0, ..., \alpha_n\}$. Let $(x_\alpha) \in U$ be an arbitrary element. For each $i \in \{0, 1, ..., n\}$ choose $\epsilon_i > 0$ so that

$$B^{\bar{d}}_{\epsilon_i}(x_{\alpha_i}) \subseteq U_{\alpha_i}.$$

and set $\epsilon = inf\{\epsilon_0, ..., \epsilon_n\} > 0$. The I-tuple (x_α) corresponds to the function $f \in X^I$ is defined by $f_\alpha = x_\alpha$ for all α. Then the ϵ-ball around f satisfies

$$(x_\alpha) \in B^\rho_\epsilon(f) = \prod_{\alpha \in I} B^{\bar{d}}_\epsilon(x_\alpha) \subseteq \prod_{\alpha \in I} U_\alpha = U$$

so U is also open in the uniform topology.

Conversely, when I is finite the open balls in the uniform metric have the form

$$B^\rho_\epsilon(f) = \prod_{\alpha \in I} B^{\bar{d}}_\epsilon(x_\alpha)$$

so are basic open sets in the product topology. Thus in this case the two topologies agree. □

Definition 31. Let $(f_n)_{n=1}^\infty$ be a sequence of functions in X^I. We say that $(f_n)_{n=1}^\infty$ *converges uniformly* (with respect to a metric on X) if the sequence converges in the uniform topology. We say that $(f_n)_{n=1}^\infty$ converges *pointwise* if it converges in the product topology.

An immediate corollary of Proposition 4.6 is:

Corollary 4.8. *Uniform convergence in X^I implies pointwise convergence. The converse holds if I is finite.*

Theorem 4.9 (Uniform Limit Theorem)**.** *Let Y be a topological space, (X, d) a metric space, $(f_n : Y \to X)_{n=1}^\infty$ a sequence of continuous functions and suppose that the f_n converge uniformly to $f \in X^Y$. Then the limit f is continuous.*

Proof. Assume for simplicity that $d = \bar{d}$ (there is no loss of generality. Why?).

Given $U \subset X$, we must show that $f^{-1}(U)$ is open in Y. It is enough to construct for every $y_0 \in f^{-1}(U)$ and open neighbourhood V such that $y_0 \in V \subseteq f^{-1}(U)$.

Choose $\epsilon > 0$ small enough so that

$$B^d_\epsilon(f(y_0)) \subset U.$$

Choose N large enough so that

$$\rho(f_N, f) < \epsilon/2,$$

or equivalently $d(f(y), f_N(y)) < \epsilon/2$ for all $y \in Y$. Set

$$V := f_N^{-1}(B^d_{\epsilon/2}(f(y_0))) \subseteq Y.$$

Then $y_0 \in V$ (because $d(f(y_0), f_N(y_0)) \leq \rho(f, f_N) < \epsilon/2$) and V is open because f_N is continuous.

For any other point $y \in V$ we have

$$\begin{aligned} d(f(y), f(y_0)) &\leq d(f(y), f_N(y)) + d(f_N(y), f(y_0)) \\ &\leq \rho(f, f_N) + \epsilon/2 \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

Thus $f(V) \subseteq U$ so $V \subseteq f^{-1}(U)$. We conclude that $f^{-1}(U)$ is open and that f is continuous. □

4.2 Metric Spaces and Continuity

For metric spaces we also have an $\epsilon - \delta$ definition of continuity.

Theorem 4.10. *Let $f : X \to Y$ be a map between metric spaces (X, d_X) and (Y, d_Y). Then f is continuous if and only if $\forall x \in X$ and $\forall \epsilon > 0$, $\exists \delta > 0$ such that*

$$f(B^{d_X}_\delta(x)) \subseteq B^{d_Y}_\epsilon(f(x))$$

Proof. Suppose f is continuous. Then for any x and ϵ, the preimage

$$U = f^{-1}(B^{d_Y}_\epsilon(f(x)))$$

is open in X. In particular, since $x \in U$, there exists an open ball $B^{d_X}_\delta(x) \subset U$. Then by definition of the preimage, $f(B^{d_X}_\delta(x)) \subseteq B^{d_Y}_\epsilon(f(x))$.

Conversely, suppose that the $\epsilon - \delta$ condition holds. Let $V \subset Y$ be open. We want to that $f^{-1}(V)$ is open. Let $x \in f^{-1}(V)$ so $f(x) \in V$. V is open so there exists $\epsilon > 0$ with $B^{d_Y}_\epsilon(f(x)) \subset V$. By $\epsilon - \delta$ there exists $B^{d_X}_\delta(x)$ such that

$$f(B^{d_X}_\delta(x)) \subseteq B^{d_Y}_\epsilon(f(x)) \subseteq V$$

thus $B^{d_X}_\delta(x) \subset f^{-1}(V)$, so $f^{-1}(V)$ is open. □

It is also possible to define continuity for metric spaces using convergent sequences. We begin with a lemma.

Lemma 4.11. *Let X be a topological space, and $A \subseteq X$. If there exists a sequence $\{x_n\}_{n=1}^\infty \subset A$ converging to $x \in X$ then x lies in the closure $\bar{A}$. The converge holds if X is metrizable.*

Proof. Suppose that $x_n \to x$ (i.e. x_n converges to x). Suppose that $x \notin \bar{A}$. Then $x \in \bar{A}^c$ is an open neighbourhood. By the definition of convergence, there exists $x_n \in (\bar{A})^c$ for some n. But $x_n \in A$, so this is a contradiction.

Conversely, suppose that X is metrizable and $x \in \bar{A}$. For each $n \in \mathbb{Z}_+$ choose $x_n \in A \cap B_{1/n}(x) \neq \emptyset$. For any open set $x \in U$, we have $x_n \in B_{1/n}(x) \cap A \subset U \cap A$ for n sufficiently large, thus $x_n \to x$ converges. □

Corollary 4.12. *Let X be a topological space and Y a metric space. Then the set of continuous function $Cont(X,Y)$ is a closed subset of the set of all functions X^Y in the uniform topology. This is not necessarily true in the product topology.*

Proof. The uniform limit theorem says that convergent subsequences contained in $Cont(X,Y)$ converge to an element of $Cont(X,Y)$ in the uniform metric topology on Y^X, so $Cont(X,Y)$ must be closed by Lemma 4.12.

On the other hand, the sequence of continuous functions $x^n : [0,1] \to \mathbb{R}$ converges in the product topology (i.e. converges pointwise) but converges to a non-continuous function, so by the Lemma $Cont([0,1],\mathbb{R})$ is not closed under the product topology on $\mathbb{R}^{[0,1]}$. □

Theorem 4.13. *If $f : X \to Y$ is a continuous map, then for every convergent sequence $x_n \to x$ in X, the sequence $(f(x_n))_{n=1}^{\infty}$ converges to $f(x)$ in Y. The converse holds if X is metrizable.*

Proof. Suppose f is continuous and $x_n \to x$ in X. For any open set $f(x) \in V \subset Y$, we have

$$
\begin{aligned}
f^{-1}(V) \text{ is open in X} \quad &\Rightarrow \quad x_n \in f^{-1}(U) \text{ for } n \text{ large} \\
&\Rightarrow \quad f(x_n) \in V \text{ for } n \text{ large} \\
&\Rightarrow \quad f(x_n) \to f(x)
\end{aligned}
$$

Now suppose that X is metrizable and $f(x_n) \to f(x)$ converges whenever $x_n \to x$ converges. To show that f is continuous we suppose that $C \subset Y$ is *closed* and prove that $f^{-1}(C)$ is closed.

Suppose $x \in \overline{f^{-1}(C)}$. Then

$$
\begin{aligned}
\exists (x_n)_{n=1}^{\infty} \subset f^{-1}(C), \text{ such that } x_n \to x \quad &\Rightarrow \quad f(x_n) \to f(x) \\
&\Rightarrow \quad f(x) \in C \\
&\Rightarrow \quad x \in f^{-1}(C)
\end{aligned}
$$

so we conclude that $\overline{f^{-1}(C)} = f^{-1}(C)$ is closed. □

4.3 Complete Metric Spaces

Definition 32. Let (X,d) be a metric space. A sequence $(x_n)_{n=1}^{\infty} \subset X$ is called a *Cauchy sequence* if $\forall \epsilon > 0, \exists N$ such that $\forall m,n > N$, $d(x_m,x_n) < \epsilon$.

Proposition 4.14. *Every convergent sequence is Cauchy.*

Proof. Suppose that the sequence $(x_n)_{n=1}^{\infty}$ converges to a point L. Then by definition, for all $\epsilon > 0$, there exists N such $d(x_n, L) < \epsilon/2$ for $n > N$. Consequently, if $m, n > N$ then

$$d(x_m, x_n) \leq d(x_m, L) + d(x_n, L) < \epsilon/2$$

as required. □

Definition 33. A metric space is called *complete* if every Cauchy sequence converges.

Example 47. $\mathbb{R} \setminus \{0\}$ is not complete in the Euclidean metric.

Example 48. The subspace $\mathbb{Q} \subset \mathbb{R}$ of rational numbers is not complete in the Euclidean metric. For instance $3, 3.1, 3.14, 3.141, ...$ converges to $\pi \notin \mathbb{Q}$.

Example 49. An open interval $(0, 1) \subset \mathbb{R}$ is not complete in the Euclidean metric.

Recall that topologically $(0, 1) \cong \mathbb{R}$. The next proposition demonstrates that completeness is a metric property rather than a topological property.

Proposition 4.15. $\mathbb{R}$ *with the Euclidean metric is complete.*

Proof. First observe that any Cauchy sequence $(x_n)_{n=1}^{\infty}$ is bounded. Set $\epsilon = 1$ then for some N, we have $x_n \in [x_N - 1, x_N + 1]$ when $n > N$. So all but a finite number of x_n lie in a bounded interval, so the full set $\{x_n | n \in \mathbb{Z}_+\}$ is bounded.

The Bolzano-Weierstrass theorem says that any bounded sequence of real numbers must contain a convergent subsequence. Suppose that L_1 and L_2 are limits of converent subsequences of $(x_n)_{n=1}^{\infty}$. For any $\epsilon > 0$ there exists N such that there exist $m, n > N$ such that $|x_m - L_1| < \epsilon$, $|x_n - L_2| < \epsilon$ and $|x_m - x_n| < \epsilon$. Thus by the triangle inequality

$$|L_1 - L_2| < |L_1 - x_m| + |x_m - x_n| + |x_n - L_2| < 3\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we deduce that $|L_1 - L_2| = 0$ so $L_1 = L_2$, so the sequence converges to a unique limit. □

Proposition 4.16. $\mathbb{R}^k$ *is complete with respect to the Euclidean metric* d.

Proof. Suppose $(\vec{x}_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}^k$, where $\vec{x}_n = (x_n^1, ...x_n^k)$. Then since $d(\vec{x}_m, \vec{x}_n) \geq |x_m^i - x_n^i|$ for any i, we also have $(x_n^i)_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$ for each $i \in \{1, ..., k\}$ and thus much converge to a limit $L^i \in \mathbb{R}$. But then the sequence $(\vec{x}_n)_{n=1}^{\infty}$ converges to

$(L^1, ..., L^k)$ pointwise, which is the same as converging in the product topology. Since we know the product topology equals the Euclidean topology, this proves that $\mathbb{R}^n$ is complete in the Euclidean metric. □

Theorem 4.17. *Let (X,d) be a complete metric space. A subset $A \subset X$ is complete with respect to the restricted metric, if and only if A is closed.*

Proof. A is closed $\Leftrightarrow$ Every convergent sequence contained in A converges to a point in $A \Leftrightarrow$ Every Cauchy sequence in A converges in A. □

Corollary 4.18. *Subspaces of $\mathbb{R}^n$ are complete in the Euclidean metric if and only if they are closed.*

Theorem 4.19. *Let (X,d) be a complete metric space, I a set. The function space X^I is complete in the uniform metric.*

Proof. Assume w.l.o.g. that $d = \inf\{d,1\}$ is bounded. Let

$$\rho(f,g) = \sup\{d(f(\alpha), g(\alpha)) | \alpha \in I\}$$

denote the uniform metric for $f, g \in X^I$.

Let $(f_n : I \to X)_{n=1}^{\infty}$ be a Cauchy sequence in X^I. Then

$$\rho(f_m, f_n) \geq d(f_m(\alpha), f_n(\alpha)), \qquad \forall \alpha \in I$$

So for every $\alpha \in I$, the sequence $(f_n(\alpha))_{n=1}^{\infty}$ is Cauchy in X, thus convergent. Define $f \in X^I$ by

$$f(\alpha) = \lim_{n\to\infty} f_n(\alpha).$$

We will show that $f_n \to f$ uniformly.

Given $\epsilon > 0$, choose N large enough so that for $m, n > N$,

$$\rho(f_m, f_n) < \epsilon/2.$$

For any $\alpha \in I$ and $m > N$ we have:

$$\begin{aligned} d(f_m(\alpha), f(\alpha)) &\leq \limsup_{n\to\infty} \big(d(f_m(\alpha), f_n(\alpha)) + d(f_n(\alpha), f(\alpha))\big) \\ &\leq \limsup_{n\to\infty} \big(\rho(f_m, f_n)) + \lim_{n\to\infty} (d(f_n(\alpha), f(\alpha))\big) \\ &\leq \epsilon/2 + 0 < \epsilon \end{aligned}$$

Example 50. For any set I, the space $\mathbb{R}^I$ is complete in the uniform Euclidean metric.

□

Corollary 4.20. *Let X be a topological space and Y a complete metric space. Then $Cont(X,Y)$ is complete in the uniform metric.*

Proof. Since $Cont(X,Y)$ is closed in the complete metric space Y^X, it is also complete. □

4.4 Completing Metric Spaces

Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is called an *isometry* if for all $a, b \in X$,

$$d_X(a, b) = d_Y(f(a), f(b)).$$

Proposition 4.21. *Isometries are injective.*

Proof. Suppose that $f(a) = f(b)$. Then $d_Y(f(a), f(b)) = d_X(a, b) = 0$, so $a = b$. □

Consequently, an isometry f restricts to an isomorphism with a metric subspace of Y

$$X \cong f(X) \subseteq Y.$$

Definition 34. An isometry $f : X \hookrightarrow Y$ is called a *completion* if Y is complete and the closure $\overline{f(X)} = Y$ (i.e. $f(X)$ is *dense* in Y).

Theorem 4.22. *For any metric space (X, d), a completion exists and is unique (up to unique bijective isometry). We denote by $(\bar{X}, d)$ the* completion of *(X, d).*

Proof. Given any completion $f : X \hookrightarrow Y$, there is an induced map

$$L : Cau(X) \to Y, \qquad L((x_n)_{n=1}^{\infty}) = \lim_{n\to\infty} f(x_n)$$

where $Cau(X)$ is the set of Cauchy sequences in X. Observe that this map is well defined, because Y is complete and it is surjective because $f(X)$ is dense so every point of Y is the limit of some Cauchy sequence in $f(X)$.

Idea: Construct $\bar{X} = Cau(X)/\sim$ for an equivalence relation $\sim$ such that L descends to an isomorphism

$$(Cau(X)/\sim) \cong Y.$$

It is helpful in the proof to introduce a new definition. A *premetric* on a set S is a map $D : S \times S \to$ that satisfies all the conditions of a metric, except that $D(a, b) = 0$ does *not* imply that $a = b$.

Lemma 4.23. *Let (S, D) be a premetric space. The relation*

$$a \sim_D b \Leftrightarrow D(a, b) = 0$$

is an equivalence relation on S. The premetric descends to metric on the quotient set $S/\sim_D$. That is

$$\bar{D} : (S/\sim_D) \times (S/\sim_D) \to \mathbb{R}$$

defined by $\bar{D}([a], [b]) = D(a, b)$ is a well-defined metric.

Proof. First we show that $\sim_D$ is an equivalence relation. The only thing that is not immediate is transitivity. Suppose that $a \sim_D b \sim_D c$ so $D(a, b) = D(b, c) = 0$. Then by the triangle inequality

$$D(a, c) \leq D(a, b) + D(b, c) = 0 + 0 = 0,$$

and $a \sim_D c$ and transitivity holds.

Next, we show that $\bar{D}$ is well defined. Suppose that $a \sim_D b$ so that $[a] = [b] \in S/\sim$. Then

$$D(a, c) \leq D(a, b) + D(b, c) = D(b, c).$$

Similarly we get $D(b, c) \leq D(a, c)$, so we conclude that $D(a, c) = D(b, c)$. Thus the definition $\bar{D}([a], [c]) = D(a, c)$ is independent of the choice of representative a, c.

That $\bar{D}$ possesses all the properties of a metric now follows immediately from the fact that D is a premetric. For example, the triangle inequality:

$$\bar{D}([a], [c]) = D(a, c) \leq D(a, b) + D(b, c) = \bar{D}([a], [b]) + \bar{D}([b], [c])$$

□

Returning to $Cau(X)$, we claim that the map $D : Cau(X) \times Cau(X) \to \mathbb{R}$ by $D((x_n), (y_n)) = \lim_{n\to\infty} d(x_n, y_n)$ is a premetric on $Cau(X)$.

First must show that D is well-defined. Since (x_n) and (y_n) are Cauchy sequences, there exists N so that if $m, n > N$ we have $d(x_m, x_n) < \epsilon$ and $d(y_n, y_m) < \epsilon$. Therefore by the triangle inequality,

$$d(x_m, y_m) \leq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m) < \epsilon + d(x_n, y_n).$$

Similarly, $d(x_n, y_n) < d(x_m, y_m) + \epsilon$ so $|d(x_m, y_m) - d(x_n, y_n)| < \epsilon$ and $(d(x_n, y_n))_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$ and must converge to $D((x_n), (y_n))$.

Now we verify that D is a premetric. Certainly $D \geq 0$, $D((x_n), (x_n)) = \lim(0) = 0$ and is symmetric. To prove triangle inequality,

$$D((x_n), (y_n)) = \lim_{n\to\infty} d(x_n, y_n) \leq \lim_{n\to\infty} \Big(d(x_n, z_n) + d(z_n, y_n)\Big) = D((x_n), (z_n)) + D((z_n), (y_n)).$$

Set $\bar{X} := (Cau(X)/\sim_D)$. By the Lemma, D descends to a metric $\bar{D}$ on $\bar{X}$.

The map $i : X \hookrightarrow \bar{X}$ sending x to the constant sequence $(x)_{n=1}^{\infty}$ is an isometry and any point $[(x_n)] \in \bar{X}$ is the limit of a sequence $[i(x_n)]$ in $i(X)$, so $i(X)$ is dense.

It only remains to prove that $\bar{X}$ is complete. It is simpler to prove a more general result.

Lemma 4.24. *Let (S, d) be a metric space and let $A \subset S$ be a dense subspace, so the closure of A is S. If every Cauchy sequence in A converges in S, then S is complete.*

Proof. Choose $(s_n)_{n=1}^{\infty}$ a Cauchy sequence in S. Since A is dense, we can construct another sequence $(a_n)_{n=1}^{\infty}$ in A such that $d(s_n, a_n) < 1/n$. For any $\epsilon > 0$ choose N so that if $m, n > N$,

$$d(s_m, s_n) < \epsilon/2, \qquad \text{and} \qquad 1/N < \epsilon/4$$

Then

$$d(a_m, a_n) \leq d(a_m, s_m) + d(s_m, s_n) + d(s_n, a_n) < \epsilon/4 + \epsilon/2 + \epsilon/4 = \epsilon$$

so (a_n) is Cauchy, hence it converges to a limit $L \in S$. So for any $\epsilon > 0$, there exists $N > 0$ so that

$$d(L, s_n) \leq d(L, a_n) + d(a_n, s_n) < \epsilon/2 + \epsilon/2$$

and so $s_n \to L$ also converges. □

Uniqueness can be proven using abstract nonsense. □

Example 51. Let $\mathbb{Q} \hookrightarrow \mathbb{R}$ inclusion of the rational numbers into the reals is the completion of $\mathbb{Q}$ under the Euclidean metric. This is in fact one way to construct the real numbers from the rationals.

Example 52. If X is any subset of $\mathbb{R}^n$ with the Euclidean metric, then the completion may be canonically identified with the closure $\bar{X} \subset \mathbb{R}^n$ (prove this).

Example 53. *p-adic numbers.* This is a completion of $\mathbb{Q}$ using a different metric. Let $p \in \mathbb{Z}$ be a prime number. Given any element $r \in \mathbb{Q}$, there is a unique integer N such that

$$r = p^N \frac{a}{b}$$

where $a, b \in \mathbb{Z}$. The *p-adic norm* is a map $\mathbb{Q} \to \mathbb{R}$ defined by

$$||r||_p = p^{-N}.$$

The *p-adic metric* on $\mathbb{Q}$ is defined by

$$d_p(r, s) = ||r - s||_p.$$

I leave it as an exercise to prove this is a metric. Observe that two rational numbers r, s are close together in this metric if their difference is highly divisible by p (or rather has many factors of p in the numerator compared to the denominator). The completion of $\mathbb{Q}$ with respect to d_p is the set $\mathbb{Q}_p$ of *p-adic numbers*.

Example 54. Let $[0, 1]$ be the unit interval and consider the set $X := Cont([0, 1], \mathbb{R})$ of continuous functions in the Euclidean topologies. For any real number p with $1 \leq p < \infty$, the L^p-norm is defined:

$$||.||_p : X \to \mathbb{R} \qquad , \qquad ||f||_p = \Big(\int_0^1 |f(t)|^p dt \Big)^{1/p}$$

The L^p-metric is

$$d_p(f, g) = ||f - g||_p$$

The completion is the space $L^p([0, 1])$ of L^p-functions.

Aside: There is a notational convention that when $p = \infty$ $||.||_p$ equals the uniform metric. It interesting to ponder why this is a good convention.

5 Connected Spaces

Definition 35. Let X be a topological space. A *separation* of X is a pair of non-empty, open sets $U, V \subset X$ satisfying

$$U \cap V = \emptyset \qquad , \qquad U \cup V = X.$$

In other words, U, V are disjoint and cover X. We say X is *connected* if no separation of X exists. We say a subset $A \subset X$ is connected is it is connected in the subspace topology.

Remark 4. Observe that for any separation U, V of X, $U = V^c$ and $V = U^c$ are also both closed. Thus we may also define a separation to be a pair of non-empty closed sets U, V satisfying $U \cap V = \emptyset$ and $U \cup V = X$.

Example 55. The subspace $X := [0,1] \cup [2,3] \subset \mathbb{R}$ is not connected in the Euclidean topology, because the pair $U = [0,1]$ and $V = [2,3]$ is a separation of X.

Proposition 5.1. *A space X is connected if and only if the only subsets of X which are both open and closed are $\emptyset$ and X.*

Proof. A subset $U \subseteq X$ is both open and closed if and only if U and U^c are open. Thus is $U \neq \emptyset, X$ then U, U^c separate X. If U, V separate X then $V = U^c$ and U is both open and closed. □

Definition 36. A subspace $X \subset \mathbb{R}$ is called *convex* if for all $a, b \in X$ the interval $[a, b] \subseteq X$.

It is not hard to show that the convex subsets of $\mathbb{R}$ are the open, closed and half-open intervals, the open and closed rays, and $\mathbb{R}$ itself.

Proposition 5.2. *Let $X \subset \mathbb{R}$ with the Euclidean topology. X is convex if and only if X is connected.*

Proof. First, assume that X is not convex. Then for some pair $a, b \in X$ there exists $r \in [a, b] \cap X^c$. Then $a \in U = (-\infty, r) \cap X$ is open and $b \in U^c = (r, \infty) \cap X$ is also open so U, U^c separate X.

Now suppose X is not connected and let $A, B \subset X$ be a separation. Choose points $a \in A$ and $b \in B$ and assume without loss of generality that $a < b$. We aim to show that $[a, b] \not\subseteq X$.

Assume that $[a, b] \subseteq X$. Then the sets $A_0 = A \cap [a, b]$ and $B_0 = B \cap [a, b]$ form a separation of $[a, b]$. Since both A_0 and B_0 are closed in $[a, b]$ and $[a, b]$ is also closed in $\mathbb{R}$, it follows that A_0 and B_0 are closed in $\mathbb{R}$ (a closed subspace of a closed subspace is closed by Lemma 3.12).

In particular the supremum

$$c = \sup(A_0)$$

must be an element of A_0. We conclude that $c \neq b \in B_0$. The non-empty set $(c, b]$ is disjoint from A_0 so it must be contained in B_0. But since B_0 is closed, $c \in B_0$ which is a contradiction because A_0, B_0 are disjoint. Thus

$$[a,b] \nsubseteq X$$

so X is not convex.

□

The most important property of connectedness is the following.

Theorem 5.3. *Let $f : X \to Y$ be a continuous map, with X connected. Then the image $f(X)$ is connected.*

Proof. Suppose that U and V separate $f(X)$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty, open and satisfy

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$$

and

$$f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V) = f^{-1}(f(X)) = X$$

so they separate X. □

Example 56. The circle S^1 can be constructed as a quotient space $[0,1]/\{0 \sim 1\}$. The quotient map $Q : [0,1] \to S^1$ is surjective, so S^1 is connected.

Corollary 5.4 (Intermediate Value Theorem)**.** *Let $f : X \to \mathbb{R}$ be continuous with X connected. For $a, b \in X$ let $r \in [f(a), f(b)]$. Then there exists $c \in X$ for which $f(c) = r$.*

Proof. X is connected so $f(X)$ is a connected subset of $\mathbb{R}$ thus it is convex, so $r \in f(X)$. □

Lemma 5.5. *Let X be a topological space and let $\{A_\alpha\}_{\alpha \in I}$ be a collection of connected subspaces, each containing a common point $p \in A_\alpha$ for all $\alpha \in I$. Then the union $A := \bigcup_{\alpha \in I} A_\alpha$ is connected.*

Proof. Suppose U and V separate A and assume without loss of generality that $p \in U$. Then since A_α is connected, one of $U \cap A_\alpha$ and $V \cap A_\alpha$ must be empty. Since $p \in U \cap A_\alpha$ it must be that $V \cap A_\alpha = \emptyset$. But this is true for all α so $V \cap A = \emptyset$ also, which is a contradiction. □

Proposition 5.6. *Given a topological space X define a relation $\sim$ by $x_1 \sim x_2$ if there exists a connected subspace $A \subset X$ containing both x_1 and x_2. Then $\sim$ is an equivalence relation. The equivalence classes of $\sim$ are the maximal connected subspaces of X and are called the **connected components** of X.*

Proof. Clearly $\sim$ possesses both the identity and symmetry properties. It remains prove transitivity. Suppose $x \sim y$ and $y \sim z$. Then there exist connected subsets $A, B \subseteq X$ such that $x, y \in A$ and $y, z \in B$. Then by Lemma 5.5 $A \cup B$ is connected and contains x and z so $x \sim z$.

To see that the equivalence classes are connected, observe that $[x]$ equals the union of all connected subsets $A \subset X$ containing x, so by Lemma 5.5 $[x]$ is connected. □

Proposition 5.7. *Let $\{X_i\}_{i=1}^n$ be a finite collection of connected spaces. Then the product $\prod_{i=1}^n X_i$ is connected..*

Proof. By induction, it suffices to consider the case $n = 2$. Let X and Y be connected spaces and choose a fixed base point $(a, b) \in X \times Y$. The subspace $X \times \{b\} \cong X$ is connected and for each $x \in X$, the space $\{x\} \times Y \cong Y$ is connected. Applying the Lemma

$$T_x := X \times \{b\} \cup \{x\} \times Y$$

is open, because they share the common point (x, b). Since T_x contains (a, b) for every value of x, we apply the Lemma again to see that

$$X \times Y = \bigcup_{x \in X} T_x$$

is connected. □

A more general result holds.

Theorem 5.8. *Let $\{X_\alpha\}_{\alpha \in I}$ be a collection of connected spaces. Then the product space $\prod_{\alpha \in I} X_\alpha$ is connected.*

Before we prove this, we need a lemma.

Lemma 5.9. *Let $A \subseteq X$ be a connected subspace. Then the closure $\bar{A} \subseteq X$ is connected.*

Proof. Let $U, V \subseteq \bar{A}$ separate $\bar{A}$. Then either $U \cap A$ or $V \cap A$ is empty. Say $U \cap A = \emptyset$. Then $A \subset V$, so $\bar{A} \subset \bar{V} = V$ because V is closed in $\bar{A}$ hence closed in X. This contradicts U, V being a separation of $\bar{A}$. □

Proof of Theorem 5.8. Choose a fixed basepoint $(a_\alpha) \in \prod_{\alpha \in I} X_\alpha$. Given a finite subset $K \subset I$ define

$$X_K := \{(x_\alpha) \in \prod_{\alpha \in I} X_\alpha \mid x_\alpha = a_\alpha \text{ if } \alpha \notin K \}$$

The subset X_K is homeomorphic to a finite product of connected spaces, so it is connected and it contains the basepoint. Applying Lemma 5.5 we learn that the subspace

$$Y := \bigcup_{\text{finite } K \subset I} X_K$$

is connected. By Lemma 5.9 it suffices to prove that Y is dense in $\prod_{\alpha \in I} X_\alpha$.

Recall that the basic open sets in the product space have the form $B = \prod_{\alpha \in I} U_\alpha$, where the U_α are nonempty, open and $U_\alpha = X_\alpha$ for all α outside of finite subset $K \subset I$. But then

$$Y \cap B \supset X_K \cap B \neq \emptyset,$$

so Y intersects every basic open set B, and hence every non-empty open set. Thus $\bar{Y} = \prod_{\alpha \in I} X_\alpha$ completing the proof. □

Example 57. $\mathbb{R}^I$ and $[0, 1]^I$ are connected in the product topology for any I. Quotients of these spaces are connected.

5.1 Path-Connected Spaces

Definition 37. Let X be a topological space. A *path* in X is a continuous map $f : [0,1] \to X$. We say that the path f joins $a, b \in X$ if $f(0) = a$ and $f(1) = b$. A space X is called *path-connected* if every two points $a, b \in X$ can be joined by a path.

Proposition 5.10. *The relation*

$$a \sim b \qquad \text{if } a \text{ can be joined to } b$$

is an equivalence relation. The equivalence classes of $\sim$ are called the ***path-components*** *of X and are equal to the maximal path-connected subspaces of X.*

Proof. Clearly $a \sim a$ for any $a \in X$. If $f(t) : [0,1] \to X$ joins a to b then $f(1-t)$ joins b to a, so $\sim$ is symmetric. Now suppose that f_1 joins a to b and f_2 joins b to c. Define the concatenation

$$g(t) := \begin{cases} f_1(2t) & \text{if } t \in [0, 1/2] \\ f_2(2t-1) & \text{if } t \in [1/2, 1] \end{cases}$$

Then g restricts to a continuous function on both $[0, 1/2]$ and $[1/2, 1]$. These form a finite closed cover of $[0,1]$ so g is continuous and joins $g(0) = a$ to $g(1) = c$. It follows that $\sim$ satisfies transitivity.

That the path-components are equal to the maximal path-connected subsets follows immediately from $\sim$ being an equivalence relation. □

Theorem 5.11. *If a space X is path-connected, then it is connected.*

Proof. Suppose that X is not connected. Then there exists a separation U, V of X (so U, V are open, nonempty, disjoint and cover X). Choose $a \in U$ and $b \in V$ and suppose there exists a path $f : [0,1] \to X$, with $f(0) = a$ and $f(1) = b$. Then $f^{-1}(A)$ and $f^{-1}(B)$ must separate $[0,1]$. But this is a contradiction, because $[0,1]$ is connected. □

This is useful because it is often easier to verify that the space is path-connected than to prove it is connected. For example:

Example 58. A subset $A \subset \mathbb{R}^I$ is convex is every pair $a, b \in A$ is joined by the straight line path $f(t) := (1-t)a + tb$. Clearly every convex set is path connected, so it is also connected. This in particular includes, the n-disk $D^n := \{\vec{x} \in \mathbb{R}^n | ||\vec{x}|| \leq 1\}$.

The converse of Theorem 5.11 is not true however. Some connected spaces are not path-connected.

Example 59. *The topologist's sine curve.* Consider the continuous function

$$f : (0,1] \to \mathbb{R}, \qquad f(x) = cos(\pi/x).$$

The graph G of f is the image in $\mathbb{R}^2$ the continuous map $x \mapsto (x, f(x))$.

Since $(0, 1]$ is connected, it follows from Theorem 5.3 that G is also connected. By Lemma 5.9, the closure

$$\bar{G} = G \cup (\{0\} \times [-1, 1])$$

is also connected. However $\bar{G}$ is not path-connected.

Proof. Suppose there is a path $f : [0, 1] \to \bar{G}$ joining $(0, 0)$ to $(1, 0)$. We may write f in coordinates $f(t) = (x(t), y(t))$ with x, y both continuous real-valued functions. Let

$$c = \sup\{t \in [0, 1] \mid x(t) = 0\}.$$

Then $x(c) = 0$ (thus $c < 1$) and $x(t) > 0$ for all $t \in (c, 1]$. For every $\delta > 0$, there is a sufficiently large integer N such that

$$\frac{1}{N}, \frac{1}{N+1} \in [x(c), x(c + \delta))$$

Consequently, for every $\delta > 0$

$$y([c, c + \delta)) \supseteq \{cos(N\pi), cos((N + 1)\pi)\} = \{\pm 1\}$$

so $y(t)$ is not continuous at $t = c$. Contradiction. □

Observe that some results about path-connected spaces are much easier to prove than for connected spaces.

Theorem 5.12. *The product of any collection $\{X_\alpha\}_{\alpha \in I}$ of path-connected spaces is path-connected.*

Proof. Suppose $\bar{a} := (a_\alpha)$ and $\bar{b} := (b_\alpha)$ are two points in $\prod_{\alpha \in I} X_\alpha$. For each $\alpha \in I$, there exists a path $f_\alpha : [0, 1] \to X_\alpha$ joining a_α to b_α. Then the product map

$$(f_\alpha)_{\alpha \in I} : [0, 1] \to \prod X_\alpha$$

joins $\bar{a}$ to $\bar{b}$. □

Theorem 5.13. *The continuous image of a path connected space is path connected.*

Proof. Suppose $\phi : X \to Y$ is continuous and X is path connected. For any pair of points $\phi(a)$ and $\phi(b)$ in $\phi(X)$, choose a path $f : [0, 1] \to X$ joining a to b. Then the composition $\phi \circ f$ joins $\phi(a)$ and $\phi(b)$. □

Proposition 5.14. *Let $\{A_\alpha\}_{\alpha\in I}$ be a collection of path-connected subspaces of a space X, all containing a common base point p. Then the union $A := \bigcup_{\alpha\in I} A_\alpha$ is path-connected.*

Proof. If $x \in A_\alpha \subset A$, then x can be joined to p in A_α hence also in A. Thus every point in A can be joined to p and since joinability is an equivalence relation, we conclude that A is path-connected. □

Example 60. Suppose that X is a (path-)connected space. If $\hat{X}$ is constructed by gluing an n-cell onto X, for $n \geq 1$, then $\hat{X}$ is also (path-)connected.

Proof. The space $\hat{X}$ is constructed as a quotient space

$$\hat{X} := (X \coprod D^n)/\sim,$$

where $\sim$ is the equivalence relation generated by $\phi(p) \cong p$, where $\phi : S^{n-1} \to X$ is a gluing map and $S^{n-1} \subset D^n$ is the boundary sphere.

If $Q : X \coprod D^n \to \hat{X}$ is the quotient map, then the image sets $Q(X)$ and $Q(D^n)$ continuous images of (path-)connected space, and thus are (path-)connected. For any point $p \in S^{n-1}$, $Q(p)$ lies in the intersection of $Q(X)$ and $Q(D^n)$, so

$$\hat{X} = Q(X) \cup Q(D^n)$$

is a union of (path-)connected spaces sharing a common base point, thus is (path-)connected. □

6 Compactness

Definition 38. A collection $\mathcal{A}$ of subsets of a space X is called a *covering* if $\bigcup_{A\in\mathcal{A}} A = X$. More generally, if $Y \subset X$, then we say $\mathcal{A}$ is a *covering of* Y if $\bigcup_{A\in\mathcal{A}} A \supseteq Y$. An *open covering* is a covering by open sets.

Definition 39. A topological space X is *compact* if every open covering of $\mathcal{A}$ contains a finite subcollection $\{A_1, ..., A_n\} \subseteq \mathcal{A}$ that also covers X. We call $\{A_1, ..., A_n\}$ as *finite subcover.*

Definition 40. Any topology on a finite set is compact, because any open cover is already finite.

Example 61. Suppose that $(x_n)_{n=1}^{\infty}$ is a convergent sequence in a space X, with limit $x_n \to L$. Then the subspace $\{x_n | n \in \mathbb{Z}_+\} \cup \{L\}$ is a compact subset of X, because any open set containing L contains all but finitely many elements in the sequence.

Example 62. $\mathbb{R}$ is not compact in the Euclidean topology, because $\mathcal{A} := \{(n, n+2) | n \in \mathbb{Z}\}$ is an open cover with no finite subcover.

Example 63. The half open interval $(0, 1] \subset \mathbb{R}$ is not compact, because $\mathcal{A} := \{(\frac{1}{n}, 1] | n \in \mathbb{Z}_+\}$ has no finite subcover.

Theorem 6.1. *Every closed and bounded interval $[a, b] \subset \mathbb{R}$ is compact.*

Proof. Let $\mathcal{A}$ be an open cover of $[a, b]$. Define a subset

$$S := \{\lambda \in [a, b] \text{ for which } [a, \lambda] \text{ has a finite subcover in } \mathcal{A}\ \}$$

Our aim is to prove that $b \in S$. First observe that $a \in S$, because $[a, a] = \{a\}$ is a finite set. Thus S is nonempty, and is bounded above, so $c := \sup(S)$ exists.

Claim: $c \in S$.

Proof. If $c = a$ then this is clear. Now suppose $a < c$. Choose $U \in \mathcal{A}$ such that $c \in U$. Because c is the supremum of S, there exists

$$d \in (a, c] \cap S \neq \emptyset,$$

and we conclude that $[a, d]$ has a finite subcover $\{A_1, ..., A_n\} \subset \mathcal{A}$. But then $\{A_1, ..., A_n, U\}$ is a finite cover of $[a, c]$. □

Claim: $c = b$.

Proof. Suppose $c < b$. If $\{A_1, ..., A_n\}$ covers $[a, c]$, it also covers $[a, c + \epsilon]$ for $\epsilon > 0$ small $\Rightarrow c + \epsilon \in S$ which is a contradiction. □

□

Theorem 6.2. *Every closed subset of a compact space is compact.*

Proof. Let X be compact, $Y \subseteq X$ closed and let $\mathcal{A}$ be an open covering of Y. By definition of the subspace topology, every open covering of Y has the form

$$\mathcal{A} := \{U \cap Y | U \in \tilde{\mathcal{A}}\}$$

where $\tilde{\mathcal{A}}$ is a collection of open sets in X covering Y. Then $\tilde{\mathcal{A}} \cup \{X \setminus Y\}$ is an open covering of X, so it contains a finite subcovering $\{U_1, ..., U_n\}$, and $\{U_i \cap Y\}_{i=1}^n$ forms a finite cover of Y contained in $\mathcal{A}$. □

Example 64. Every closed and bounded subspace of $\mathbb{R}$ is compact, because is also a closed subset of the compact space $[-N, N]$ for some large N.

Theorem 6.3. *Every compact subspace of a Hausdorff space is closed.*

Proof. Let X be Hausdorff and $Y \subseteq X$ compact. If $Y = X$ then Y is closed and the result is immediate.

So suppose Y is a proper subset of X. Fix a point $p \in X \setminus Y$. For each $y \in Y$, there exist disjoint open neighbourhoods U_y, V_y containing y and p respectively. Construct in this fashion an open cover $\{U_y | y \in Y\}$ of Y. Since Y is compact, there exists a finite subcover $\{U_{y_1}, ..., U_{y_n}\}$, such that

$$\bigcup_{i=1}^n U_{y_i} \supseteq Y.$$

From this we conclude that the finite intersection

$$\bigcap_{i=1}^{n} V_{y_i}$$

is an open neighbourhood of p contained in Y^c. Such a neighbourhood exists for all $p \in Y^c$ so Y^c is open, and Y is closed. □

Theorem 6.4. *The product of finitely many compact spaces is compact*

Before proving this we must establish a lemma.

Lemma 6.5 (The Tube Lemma)**.** *Let X and Y be spaces, with Y compact. Suppose that an open subset $N \subseteq X \times Y$ contains a "slice" $\{x_0\} \times Y$. Then there exists an open nbhd $x_0 \in W \subseteq X$ such that $W \times Y \subset N$.*

Proof. Recall that the basic open sets in $X \times Y$ have the form $U \times V$ for open sets $U \subseteq X$ and $V \subseteq Y$. Let

$$\mathcal{A} := \{U \times V \text{ basic } | U \times V \subseteq N\}.$$

This forms an open cover of N hence also of $\{x_0\} \times Y$. Since $\{x_0\} \times Y$ is compact, there is a finite subcover $\{U_i \times V_i\}_{i=1}^{n}$ (we insist that each of these actually intersects the slice). Set $W = \bigcap_{i=1}^{n} U_i$. Then W is an open neighbourhood of x_0 and

$$W \times Y \subseteq \bigcup_{i=1}^{n} U_i \times V_i \subseteq N$$

proving the lemma. □

Proof of Theorem. It is enough to show that the product of two compact sets is compact. Let X, Y be compact let $\mathcal{A}$ be an open cover of $X \times Y$. For each $x \in X$, there exists a finite subcover $\mathcal{A}_x$ of the slice $\{x\} \times Y$. By the tube lemma, there exists an open neighbourhood $x \in W_x \subseteq X$ such that

$$W_x \times Y \subseteq \bigcup_{A \in \mathcal{A}_x} A.$$

The collection $\{W_x | x \in X\}$ is an open cover of X so it contains a finite subcover $\{W_{x_i}\}_{i=1}^{n}$. Then,

$$X \times Y = \bigcup_{i=1}^{n} W_{x_i} \times Y \subseteq \bigcup_{i=1}^{n} \Big(\bigcup_{A \in \mathcal{A}_{x_i}} A \Big)$$

so, $\mathcal{A}_{x_1} \cup ... \cup \mathcal{A}_{x_n} \subset \mathcal{A}$ is a finite subcovering of $X \times Y$, so $X \times Y$ is compact. □

Theorem 6.6. *Let X be a subspace of $\mathbb{R}^n$ with the Euclidean topology. Then X is compact if and only if X is closed and bounded.*

Proof. Exercise on problem set 5. □

Now that we have constructed a large class of compact sets: What are they good for? One of the most useful properties of compact sets is the following.

Theorem 6.7. *The image of a compact space under a continuous map is compact.*

Proof. Suppose X is compact and $f : X \to Y$ is a continuous map. Let $\mathcal{A}$ be an open cover of $f(X) \subset Y$. Then the collection

$$f^{-1}\mathcal{A} := \{f^{-1}(U) | U \in \mathcal{A}\}$$

is an open cover of X. Since X is compact, there is a finite subcover $\{f^{-1}(U_i)\}_{i=1}^n$ of X, but this implies that $\{U_i\}_{i=1}^n \subset \mathcal{A}$ is a finite subcovering of $f(X)$, so $f(X)$ is also compact. □

Corollary 6.8 (Extreme Value Theorem)**.** *Let $f : X \to \mathbb{R}$ be a continuous map with X compact and non-empty. Then there exists $x_i, x_s \in X$ such that*

$$f(x_i) = \inf(f(X)), \qquad f(x_s) = \sup(f(X))$$

Proof. The image $f(X) \subset \mathbb{R}$ is compact, so it is closed and bounded. It follows that $\inf(f(X))$ and $\sup(f(X))$ both exist and are contained in $f(X)$, so the result follows. □

Theorem 6.9. *Let $f : X \to Y$ be bijective and continuous, where X is compact and Y is Hausdorff. Then f is a homeomophism.*

Proof. In remains to show that the inverse map $f^{-1} : Y \to X$. We do this by proving that if $C \subseteq X$ is closed, then $(f^{-1})^{-1}(C) = f(C)$ is closed in Y.

Let $C \subset X$ be closed. Since X is compact, it follows that C is compact by Theorem 6.2. Since f is continuous, by Theorem 6.7 we know that $f(C)$ is compact. By Theorem 6.3 it follows that $f(C)$ is closed. □

6.1 Compact Metric Spaces

Definition 41. Let (X, d) be a metric space, and let $A \subseteq X$ with $A \neq \emptyset$. For $p \in X$ define the distance from p to A by

$$d(p, A) = \inf\{d(p, a) | a \in A\}$$

Proposition 6.10. *For fixed, nonempty $A \subseteq X$, the map $p \mapsto d(p, A)$ is a continuous map from X to $\mathbb{R}$.*

Proof. Both $\mathbb{R}$ and X are metric spaces, so we may use an $\epsilon - \delta$ proof. We will prove that for any $p, q \in X$, $|d(p, A) - d(q, A)| \leq d(p, q)$, so setting $\epsilon = \delta$ will suffice.

By the triangle inequality, for all $a \in A$,

$$d(p, A) \leq d(p, a) \leq d(p, q) + d(q, a)$$

Taking infimum over $a \in A$, we have

$$d(p, A) \leq d(p, q) + d(q, A) \Rightarrow d(p, A) - d(q, A) \leq d(p, q)$$

Reversing the roles of p and q gives $d(q, A) - d(p, A) \leq d(p, q)$, so $|d(p, A) - d(q, A)| \leq d(p, q)$ as desired.

□

Lemma 6.11 (Lebesgue number Lemma)**.** *Let $\mathcal{A}$ be an open covering of a compact metric space X. There exists $\delta > 0$, called the **Lebesgue number**, such that for all $p \in X$, the open ball $B_\delta(p)$ is contained in some $U \in \mathcal{A}$.*

Proof. If $X \in \mathcal{A}$ then any value of $\delta > 0$ will do. So we assume that $X \notin \mathcal{A}$. Because X is compact, we may choose a finite subcover $\{U_1, ..., U_n\} \subseteq \mathcal{A}$. Let $C_i = X \setminus U_i \neq \emptyset$ and define a function

$$f : X \to \mathbb{R}, \qquad f(p) = \frac{1}{n} \sum_{i=1}^{n} d(p, C_i)$$

so $f(p)$ is the *average distance from p to the sets C_i*. Note that f is continuous because it is a linear combination of continuous real valued functions (follows from Proposition 6.10 and a homework problem).

The function f is useful for us, because $d(p, C_i) \geq f(p)$ for some i, so that $B_{f(p)}(p) \subseteq U_i$. Thus to prove the Lemma it is enough to find a positive lower bound for f.

Claim: $f(p) > 0$ for all $p \in X$.

Proof. Choose U_i containing p. Since U_i is open, $B_\epsilon(p) \subset U_i$ for some $\epsilon > 0$. Therefore $d(p, C_i) \geq \epsilon$, so $f(p) \geq \epsilon/n > 0$. □

By the extreme value theorem, there exists $p_0 \in X$ such that $\inf\{f(p)|p \in X\} = f(p_0) > 0$. Setting $\delta = f(p_0)$ completes the argument.

□

Example 65. Consider the open cover of the open interval $(0, 1)$ by $\mathcal{A} := \{(\frac{1}{2^{n+2}}, \frac{1}{2^n})|n \in \mathbb{Z}_{\geq 0}\}$. This cover admits no Lebesgue number, because each $\epsilon \in (0, 1)$ is only contained in any intervals of length less than or equal to 3ϵ.

Definition 42. A map $f : X \to Y$ between metric spaces is **uniformly continuous**, if for all $\epsilon > 0$, there exists $\delta > 0$ such that for any $p, q \in X$,

$$d_X(p, q) < \delta \Rightarrow d_Y(f(p), f(q)) < \epsilon.$$

Example 66. The map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is *not* uniformly continuous with respect to the Euclidean metric. For any fixed $\epsilon > 0$, the required value of $\delta > 0$ gets smaller and smaller as $x \to \infty$ and the slope of the graph gets steeper.

Theorem 6.12. *If X is compact and X, Y are metric spaces, then any continuous map $f : X \to Y$ is uniformly continuous.*

Proof. Given $\epsilon > 0$, define an open cover $\mathcal{A} := \{f^{-1}(B_{\epsilon/2}(y)) | y \in Y\}$. Let $\delta > 0$ be the Lebesgue number for $\mathcal{A}$. Then $d_X(p, q) < \delta$ implies that $p, q \in B_\delta(p) \subset f^{-1}(B_{\epsilon/2}(y))$ for some $y \in Y$. Thus

$$d_Y(f(p), f(q)) \leq d_Y(f(p), y) + d_Y(y, f(q)) < \epsilon/2 + \epsilon/2 = \epsilon$$

□

6.2 Compactness and Limits

Definition 43. A space X is called *limit point compact* if every infinite subset $A \subset X$ has a limit point.

Theorem 6.13. *If X is compact, then it is limit point compact.*

Proof. Suppose that X is compact can $A \subseteq X$ has no limit points. We will show that A is finite.

Since A has no limit points, we know that it contains all of its (non-existent) limit points so by ... we know that $A = \bar{A}$ is closed. Closed subspaces of compact spaces are compact, so A is compact. Since A has no limit points, for each $p \in A$, there exists an open set $U_p \subset X$, such that $U_p \cap \{p\} = \{p\}$, so the singleton subsets of A are open. The collection of singleton sets $\mathcal{A} := \{\{p\} | p \in A\}$ then forms an open cover, with no proper sub-covers. Since $\mathcal{A}$ must admit a finite subcover, we conclude that A must be finite. □

The converse of Theorem 6.13 does not hold.

Example 67. Let $X = \{a, b\}$ be a two element set equipped with the trivial topology (only $\{\emptyset, X\}$ are open). Now let $\mathbb{Z}$ have the discrete topology and consider the product space $\mathbb{Z} \times X$. In this topology, any subset containing (n, a) has (n, b) as a limit point, and vice-versa. Thus *every* non-empty subset of $\mathbb{Z} \times X$ contains a limit point, so $\mathbb{Z} \times X$ is limit point compact. However, it is not compact because the open cover $\mathcal{A} := \{\{n\} \times X | n \in \mathbb{Z}\}$ contains no finite subcovers.

Observe also that the projection map $\mathbb{Z} \times X \to \mathbb{Z}$ is continuous and surjective, but $\mathbb{Z}$ is not limit point compact ($\mathbb{Z}$ is discrete so no subset has any limit points). Thus continuous images of limit point compact spaces are *not* limit point compact in general.

The preceding example explains why compactness is a more useful concept in general topology than limit point compactness. However, for metric spaces they are the same thing.

Theorem 6.14. *A metric space (X, d) is compact if and only if it is limit point compact.*

One direction has already been proven in Theorem 6.13. To prove the other direction, we require a lemma.

Lemma 6.15. *If (X, d) is limit point compact, then every sequence $(x_n)_{n=1}^{\infty}$ contains a convergent subsequence.*

Proof. Let $S := \{x_1, ...x_n, ...\}$ be the set of values of the sequence. If S is infinite, then it contains a limit point, L. If S is finite, then some value $L \in S$ occurs infinitely often. In either case, construct a subsequence $(x_{n(i)})_{i=1}^{\infty}$ converging to L recursively by

$$x_{n(i)} \in B_{\frac{1}{i}}(L) \text{ and } n(i) > n(i-1).$$

□

Proof of Theorem. Let (X, d) be a limit point compact space.

Claim: The Lebesgue Number Lemma holds for X.

Proof. Let $\mathcal{A}$ be an open cover of X. Assume for the sake of contradiction that there is no Lebesgue number for $\mathcal{A}$. That is, for every $\delta > 0$, there exists $p \in X$ such that $B_\delta(p)$ is not contained in any element of $\mathcal{A}$.

Construct a sequence, $(x_n)_{n=1}^{\infty}$ such that $B_{1/n}(x_n)$ is not contained in any element of $\mathcal{A}$. By the preceding lemma, we know that some subsequence $(x_{n(i)})_{i=1}^{\infty}$ converges to a limit L.

Choose $U \in \mathcal{A}$ containing L. Then $B_\epsilon(p) \subset U$ for some $\epsilon > 0$. If i is chosen so that $\frac{1}{n(i)} < \epsilon/2$ and $d(L, x_{n(i)}) < \epsilon/2$, then

$$B_{\frac{1}{n(i)}}(x_{n(i)}) \subseteq B_{\epsilon/2}(x_{n(i)}) \subseteq B_\epsilon(L) \subseteq U$$

which is a contradiction. □

Claim: For any $\epsilon > 0$, there exists a finite covering of X by ϵ-balls.

Proof. Assume such a cover does not exist for some $\epsilon > 0$. Construct a sequence by $x_1 \in X$, $x_2 \in X \setminus B_\epsilon(x_1)$,...., $x_n \in X \setminus \bigcup_{i<n} B_\epsilon(x_i)$, ... For every $m \neq n$, $d(x_m, x_n) \geq 2\epsilon$ so no subsequence is Cauchy, so no subsequence converges. Contradiction. □

Finally, given an open covering $\mathcal{A}$ of X with Lebesgue number $\delta > 0$, choose a finite covering by δ-balls $B_\delta(x_1), ...B_\delta(x_n)$. Each $B_\delta(x_i)$ is contained inside and open set $U_i \in \mathcal{A}$, so $\{U_1, ..., U_n\}$ is a finite subcover and X is compact.

□

6.3 The One-Point Compactification

Because compactness is such a useful property, mathematicians often like to *compactify* non-compact topological spaces.

Definition 44. A compactification of a topological space X is a compact topological space Y and a continuous embedding $i : X \hookrightarrow Y$ with dense image, $\overline{i(X)} = Y$.

Example 68 (The stupid compactification)**.** Let (X, τ_X) be an arbitrary topological space. Define a compactification,

$$Y = X \cup \{*\}$$

to be the disjoint union of X with a single point $*$, and with topology

$$\tau = \{U \subseteq X \subset Y | \ U \text{ is open in X}\} \cup \{Y\}.$$

The obvious inclusion $X \hookrightarrow Y$ is a dense embedding, and Y is compact because any open covering of Y must contain Y.

The stupid compactification is stupid, because it doesn't really tell us anything interesting about X. For example, the only continuous real valued functions on Y are the constant functions (prove this), even if X admits many interesting real-valued functions.

To get a nicer compactification, it helps to place a few mild conditions on X.

Definition 45. A topological space X is called **locally compact** if for every $p \in X$, there exists an open neighbourhood $p \in U \subseteq X$ such that the closure $\bar{U}$ is compact in the subspace topology.

Example 69. Every compact space is locally compact (set $U = X$).

Example 70. $\mathbb{R}^n$ is locally compact in the Euclidean topology: Every point is contained in an open neighbourhood of the form $(a_1, b_1) \times ... \times (a_n, b_n)$, whose closure $[a_1, b_1] \times .... \times [a_n, b_n]$ is compact.

Example 71. $\mathbb{R}^\infty = \mathbb{R}^{\mathbb{Z}_+}$ is *not* locally compact. The basic open sets have the form $(a_1, b_1) \times ... \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times ....$, which have non-compact closure, $[a_1, b_1] \times .... \times \mathbb{R} \times \mathbb{R}......$

Example 72. The subspace $\{(x, y) \in \mathbb{R}^2 \mid y < 0\} \cup \{(0, 0)\} \subset \mathbb{R}^2$ is not locally compact at $(0, 0)$.

Theorem 6.16. *Let X be a locally compact, Hausdorff space. There exists a compact Hausdorff space X_∞ such that,*

- *$X \subset X_\infty$ is a subspace.*
- *The complement $X_\infty - X$ is a single point.*

*This space X_∞ is called the **one-point compactification**, and determined up to unique isomorphism by these properties.*

Remark 5. The one-point compactification is technically only a compactification if X isn't already compact to begin with. If X is compact, then $X_\infty \cong X \coprod \{\infty\}$ (prove this).

Proof. As a set $X_\infty := X \cup \{\infty\}$. The point ∞ is called the *point at infinity*. The open sets of X_∞ come in two types:

(1) $U \subseteq X \subset X_\infty$ is open in X.

(2) $X_\infty - K$, where $K \subset X$ is compact.

This is clearly a embedding, because every open set in X is of the form $X \cap U$ for a set U of type (1), and $X \cap (X_\infty - K) = X - K$ is open in X (K is compact in a Hausdorff space, hence closed). Every open neighbourhood of ∞ is of type (2), and if X is not compact, then every such neighbourhood intersects X, so X is dense in X_∞.

Claim: X_∞ is a topological space.

Proof. We must verify the axioms.

(i) $\emptyset$ is open of type (1) and X_∞ is open of type (2).

(ii) We show that finite intersections are open. Let U, V be open sets of type (1) and $X_\infty - K$, $X_\infty - C$ open sets of type (2).

  - $U \cap V$ is open in X so it is open of type (1).
  - $U \cap (X_\infty - K) = U \cap (X - K)$ is open of type (1), because K is a compact subset of a Hausdorff space, hence closed in X, so $X - K$ is open.
  - $(X_\infty - K) \cap (X_\infty - C) = X_\infty - (K \cup C)$ is open of type (2) because $K \cup C$ is a finite union of compact sets, hence compact (Homework exercise?).

(iii) First observe that an arbitrary union of open sets of type (1) is open of type (1), since it must be open in X. Now consider an arbitrary union of type (2) open sets

$$\bigcup_{\alpha \in I} (X_\infty - K_\alpha) = X_\infty - (\bigcap_{\alpha \in I} K_\alpha)$$

Now the K_α are closed in X, so $\bigcap_{\alpha \in I} K_\alpha$ is closed in X, hence compact because it is a closed subset of a compact set K_α. Therefore $\bigcup_{\alpha \in I}(X_\infty - K_\alpha)$ is open on type (2).

An arbitrary union of open sets may be expressed as a union of type (1) sets unioned with a union of type (2) sets. By what we have shown, it is enough to show that if U is type (1) and $X_\infty - K$ is type (2) then

$$U \cup (X_\infty - K) = X_\infty - (K \cap (X - U))$$

is open of type (2), because $(X - U)$ is closed in X, so $K \cap (X - U)$ is a closed subset of a compact space, hence compact. □

Claim: X_∞ is compact.

Proof. Let $\mathcal{A}$ be an open covering of X_∞. Choose an open set $V \in \mathcal{A}$ containing ∞ must be of type (2). Consequently $K := X_\infty - V$ is a compact set. Choose a finite subcovering $\{U_1, ..., U_n\}$ of K, then $\{U_1, ..., U_n, V\}$ is a finite covering of X_∞. □

Claim: X_∞ is Hausdorff.

Proof. Certainly any two points in X are separated by open sets of type (1), because X was Hausdorff to begin with. It remains to show that ∞ can be separated from any other point $p \in X$. Because X is locally compact, there exists some open neighbourhood $p \in U \subseteq X$, such that $\bar{U}$ is compact. Thus U and $X_\infty - \bar{U}$ separate p and ∞. □

We leave the verification that X_∞ is unique up to isomorphism as an exercise.

□

Example 73. $(\mathbb{R}^n)_\infty \cong S^n$.

Example 74. The cone satisfying $x^2 + y^2 = z^2$ in $\mathbb{R}^3$ has compactification: ....

Proposition 6.17. *Let $A \subseteq X$ be a subspace of a locally compact Hausdorff space X. If A is open or closed, then A is locally compact Hausdorff.*

Proof. Any subspace of a Hausdorff space is Hausdorff (simply take the separating sets in X and intersect them with A).

We first consider the case that A is open. Then A is also open in X_∞ so $X_\infty - A$ is closed, hence compact. Choose a point $a \in A$. For every point $p \in X_\infty - A$, choose separating open sets $a \in U_p$ and $p \in V_p$ with $U_p \cap V_p = \emptyset$. The collection

$$\mathcal{A} := \{V_p | p \in X_\infty - A\}$$

is an open cover of the compact set $X_\infty - A$, so we may choose a finite open cover $\{V_{p_1}, ..., V_{p_n}\}$. The intersection $U := \cap_{i=1}^n U_{p_i} \subseteq A$ is open in X_∞, hence also in A. The closure $\bar{U}$ in X_∞ is compact and satisfies

$$\bar{U} \subseteq \cap_{i=1}^n \bar{U}_i \subseteq A$$

so $\bar{U} \cap A = \bar{U}$ is also the closure of U as a subset of A, thus A is locally compact.

Now consider the case that A is closed. Then $A \cup \{\infty\}$ is closed in X_∞, hence is compact Hausdorff. Then $A \subset A \cup \{\infty\}$ is an open subset of a compact Hausdorff space, so we conclude that it must be locally compact.

□

6.4 Compact-Open Topology

Let X and Y be topological spaces, and let $Cont(X,Y)$ be the set of continuous maps $f : X \to Y$. Given subsets $K \subseteq X$ and $U \subset Y$, define

$$S(K,U) := \{f \in Cont(X,Y) | f(K) \subseteq U\}$$

Definition 46. The *compact-open topology* on $Cont(X,Y)$ is the topology generated by the subbasis $\{S(K,U)|\ K \text{ is compact and } U \text{ is open}\}$

Example 75. Let $X = *$ be the one-element space. Then there is a natural bijection

$$Cont(*,Y) \cong Y.$$

This bijection is a homeomorphism if $Cont(*,Y)$ is endowed with the compact-open topology.

Example 76. More generally, if I is any set endowed with the discrete topology, then every map from I to Y is continuous, so there is a canonical bijection

$$Cont(I,Y) \cong Y^I \cong \prod_{i \in I} Y.$$

This bijection is a homeomorphism between the compact-open topology on $Cont(I,Y)$ and the product topology on Y (Homework exercise).

Proposition 6.18. *For general topological spaces, the compact-open topology on $Cont(X,Y)$ is (non-strictly) finer than the product topology (a.k.a. the pointwise convergence topology). If Y is metrizable, then the compact-open topology is (non-strictly) coarser than the uniform topology.*

Proof. Let $\tau_p, \tau_{co}, \tau_u$ be the product, compact-open and uniform topologies respectively.

(1) $\tau_p \subseteq \tau_{co}$**:**

Recall that τ_p is the subspace topology for the inclusion

$$Cont(X,Y) \subseteq Y^X \cong \prod_{p \in X} Y.$$

The sub-basis of open sets for the product topology consist of preimages $\pi_p^{-1}(U)$, where $\pi_p : \prod_{p \in X} Y \to Y$ is projection onto the pth factor and $U \subset Y$ is open. In our new notation, $\pi_p^{-1}(U) = S(\{p\}, U)$, which is open in τ_{co}. So all the sub-basis open sets in τ_p are open in τ_{co}, so $\tau_p \subset \tau_{co}$.

(2) $\tau_{co} \subseteq \tau_u$**:**

It will suffice to show that for $K \subseteq X$ compact and $U \subseteq Y$ open, that $S(K,U) \in \tau_u$. Suppose for notational simplicity that the metric on Y, $d = \min\{d, 1\}$ is already bounded. Let ρ be the uniform metric on $Cont(X,Y)$:

$$\rho(f,g) = \sup\{d(f(x), g(x))|\ x \in X\}$$

Choose a element $f \in S(K,U)$ and let $C := U^c = Y - U$. Define a function $\mu : K \to \mathbb{R}$ by $\mu(x) := d(f(x), C)$. Then μ is continuous, because it is to composition of f with the continuous function $d(-,C) : Y \to \mathbb{R}$. Since $f(K) \subseteq U = C^c$, we know that $\mu(x) > 0$ for all x in K. Since K is compact, we deduce from the extreme value theorem that there exists $\delta > 0$, such that if $\rho(f,g) < \delta$, then $g(K) \subset U$. In particular,

$$B^{\rho}_{\delta}(f) \subseteq S(K,U)$$

so $S(K,U)$ is open in τ_u. □

7 Advanced Material

7.1 Separation Axioms

Let X be a topological set, and $A, B \subset X$ a pair of disjoint subsets. We say that A, B can be **separated** in X if there exist open sets, $U, V \subseteq X$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

Definition 47. Let X be a topological space for which *every singleton set is closed*. (We say all the points of X are closed).

- X is called **Hausdorff** if every distinct of points $p, q \in X$ can be separated.
- X is called **Regular** if every closed set A can be separated from any point $p \notin A$.
- X is called **Normal** if every pair of disjoint closed sets $A, B \subset X$ can be separated.

It is clear that each of these conditions is stronger than the last. That is

$$\text{X is normal} \Rightarrow \text{X is regular} \Rightarrow \text{X is Hausdorff}$$

We will be most interested in the normal condition in what follows. Many familiar spaces are normal.

Proposition 7.1. *Metric spaces are always normal.*

Proof. Let $A, B \subseteq X$ be a disjoint pair of closed sets in a metric space (X, d). For each point $a \in A \subseteq B^c$, there exists an $\epsilon(a) > 0$ such that $B_{\epsilon(a)}(a)$ is disjoint from B. Similarly, for each $b \in B$, there exists $\epsilon(b) > 0$ such that $B_{\epsilon(b)}(b)$ is disjoint from A. Define

$$U := \bigcup_{a \in A} B_{\epsilon(a)/3}(a), \qquad V := \bigcup_{b \in B} B_{\epsilon(b)/3}(b).$$

Certainly U and V are open and contain A and B respectively. To see that they are disjoint, assume there exists $p \in U \cap V$. Then for some $a \in A$ and some $b \in B$, we have $p \in B_{\epsilon(a)/3}(a)$ and $p \in B_{\epsilon(b)/3}(b)$. By the triangle inequality, this implies that

$$d(a,b) < (\epsilon(a) + \epsilon(b))/3 \leq \frac{2}{3} \sup\{\epsilon(a), \epsilon(b)\}$$

which is a contradiction. □

Proposition 7.2. *Compact Hausdorff spaces are always normal.*

Proof. Exercise. □

The following equivalent condition is will prove useful.

Proposition 7.3. *A space X is normal if and only if for every pair of sets $C \subseteq U \subseteq X$ with C closed and U open, there exists another open set V with*

$$C \subseteq V \subseteq \bar{V} \subseteq U. \tag{3}$$

Proof. We have $C \subseteq U$ as in the hypothesis if and only if C and U^c are disjoint closed sets. If V satisfies equation (3), then V and $(\bar{V})^c$ separate C and U^c. Conversely, if V, W form a separation of C and U^c, then $C \subseteq V \subseteq \bar{V} \subseteq W^c \subseteq U$. □

7.2 Urysohn Lemma

The following result is of basic importance to the study of normal spaces.

Lemma 7.4 (Urysohn Lemma)**.** *Let X be a normal space and let A and B be disjoint closed subsets. Let $[a, b]$ be a closed interval in $\mathbb{R}$. There exists a continuous map $f : X \to [a, b]$ such that $f(A) = a$ and $f(B) = b$.*

Remark 6. If such a function exists for every pair A, B, then X is normal because $f^{-1}([a, a+\epsilon])$ and $f^{-1}((b-\epsilon, b])$ separates A and B. Thus we may regard the Urysohn Lemma as providing a third equivalent definition of Normality: that every pair of closed spaces be "separated" by a continuous real valued function (a similar statement holds for Hausdorff and regular spaces).

Proof. For simplicity we work with $[0, 1] = [a, b]$ (no loss of generality, since $[0, 1] \cong [a, b]$).

Our basic strategy is to define f by choosing open pre-images $U_r := f^{-1}([0, r))$ for every $r \in \mathbb{Q} \cap [0, 1]$ and then extending continuously.

Step 1: Construct a family of open sets $\{U_r \mid r \in \mathbb{Q} \cap [0, 1]\}$ with the properties (*)

- $A \subseteq U_r$ and $U_r \cap B = \emptyset$ for all r.
- $r < s \Rightarrow \overline{U_r} \subseteq U_s$.

We begin by setting $U_1 = B^c$. Choose U_0 so that $A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1$ (this is possible by Proposition 7.3).

Now observe that $\mathbb{Q} \cap [0,1]$ is countable so it can be well-ordered by the natural numbers $\{r_i\}_{i=0}^{\infty}$. We choose $r_0 = 0$ and $r_1 = 1$. We construct the remaining U_{r_n} for $n \geq 2$ inductively as follows:

Let $p = \max\{r_i | i < n \text{ and } r_i < r_n\}$ and $q = \min\{r_i \mid i < n \text{ and } r_i > r_n\}$. Then select U_{r_n} so that

$$\overline{U_p} \subseteq U_{r_n} \subseteq \overline{U_{r_n}} \subseteq U_q.$$

We now extend this collection to $\{U_r | r \in \mathbb{Q}\}$ by setting $U_r = \emptyset$ if $r < 0$ and $U_r = X$ is $r > 1$. This still satisfies properties (*).

Step 2: Define a function $f : X \to \mathbb{R}$ by

$$f(x) = \inf\{r \in \mathbb{Q} | x \in U_r\}.$$

Observe that the image $f(X)$ is contained in $[0,1]$ because $U_r = \emptyset$ for $r < 0$ and $U_r = X$ for $r > 1$. Also, $f(A) = 0$, because $A \subseteq U_0$ and $f(B) = 1$ because $U_1 \cap B = \emptyset$.

It only remains to prove that f is continuous.

Claim: For $s \in \mathbb{Q}$,

$$f(x) < s \Rightarrow x \in U_s, \quad \text{so } f(U_s^c) \subseteq [s, \infty)$$

$$f(x) > s \Rightarrow x \notin U_s, \quad \text{so } f(U_s) \subseteq (\infty, s].$$

Proof. First observe that for rational numbers $r < s$ implies $U_r \subset U_s$. This implies that,

$$\{r \in \mathbb{Q} \mid x \in U_r\} = \{r \in \mathbb{Q} \mid r \geq f(x)\}$$

and the result follows easily. □

Let $(a,b) \subseteq \mathbb{R}$ be an open interval in $\mathbb{R}$. Since such intervals form a basis for the Euclidean topology on $\mathbb{R}$ it will suffice to show that $f^{-1}(a,b)$ is open.

Let $x \in f^{-1}((a,b))$ and choose rational numbers p, q so that

$$a < p < f(x) < q < b.$$

Define the open set $V = U_q \cap (\bar{U}_p)^c$. Then by the previous claim,

$$f(V) \subseteq f(U_q \cap U_p^c) \subset [p,q] \subset (a,b).$$

To see that $x \in V$, first choose a rational number $p < p' < f(x)$. Then again using the previous claim,

$$p' < f(x) < q \Rightarrow x \in U_q \cap (U_{p'})^c \subset U_q \cap (\overline{U}_p)^c$$

where in the last step have used the fact that $\overline{U}_p \subseteq U_{p'}$. Consequently, $x \in V \subset f^{-1}((a,b))$ so $f^{-1}((a,b))$ is open and f is continuous.

□

Theorem 7.5 (Urysohn's Metrization Theorem)**.** *Every normal, second countable space X is metrizable.*

Definition 48. A topological space X is called second countable if can be generated by a countable basis of open sets.

Example 77. $\mathbb{R}^n$ is second countable because it is generated by $\mathcal{B} := \{B_\epsilon(\bar{x}) \mid \bar{x} \in \mathbb{Q}^n, \epsilon \in \mathbb{Q}_{>0}\}$. Also every subspace of $\mathbb{R}^n$ is second countable, by restricting $\mathcal{B}$.

Proof. Consider the set $[0,1]^{\mathbb{Z}_+}$ equipped with the uniform metric topology,

$$\rho((x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty) = \sup\{|x_i - y_i| \mid i \in \mathbb{Z}_+\}.$$

We will construct an embedding

$$F : X \hookrightarrow [0,1]^{\mathbb{Z}_+}$$

so that $X \cong F(X)$ acquires a metric by restriction.

Step 1: There exists a countable collection of continuous maps

$$\{f_n : X \to [0,1]\}_{n=1}^\infty$$

such that

(a) $f_n(X) \subseteq [0, 1/n]$

(b) For all open $U \subseteq X$ and $p \in U$, there exists f_n such that $f_n(p) > 0$ and $f(U^c) = 0$.

Proof. Since X is second countable, there exists a countable basis $\mathcal{B} := \{B_k\}_{k=1}^\infty$. For each pair of $k, l \in \mathbb{Z}_+$ such that $\overline{B}_k \subseteq B_l$, choose a function

$$g_{k,l} : X \to [0,1]$$

such that $g_{k,l}(\overline{B}_k) = 1$ and $g_{k,l}(B_l^c) = 0$ ($g_{k,l}$ exists because X is normal).

By the definition of a basis, for every open U and $p \in U$, there exists a basic open set with $p \in B_l \subseteq U$. Because X is normal, there exists another open set which we may choose to be basic, such that

$$p \in B_k \subseteq \overline{B}_k \subseteq B_l$$

(remember points are closed) so that $g_{k,l}(p) = 1$ and $g_{k,l}(U^c) = 0$.

The collection $\{g_{k,l}\}$ is countable, so we may relabel $\{g_{k,l}\} = \{g_n\}_{n=1}^\infty$. Setting $f_n := \frac{1}{n} g_n$ completes the construction. □

We use the f_n as coordinate functions to define the map of sets

$$F = (f_n)_{n=1}^\infty : X \to [0,1]^{\mathbb{Z}_+}.$$

Note that F is by definition continuous for the product topology on $[0,1]^{\mathbb{Z}_+}$ but is not automatically continuous with respect to the uniform topology.

Step 2: F is injective.

Suppose that $x, y \in X$ and $x \neq y$. Since X is normal, it is also Hausdorff, so there exists an open set U with $x \in U$, $y \notin U$. Simply choose f_n so that $f_n(x) > 0$ and $f_n(U^c) = f_n(y) = 0$. It follows that $F(x) \neq F(y)$.

Step 3: F is continuous.

Let $p \in X$. We must show that for any $\epsilon > 0$, there exists $p \in U \subseteq X$ open, such that $x \in U \Rightarrow \rho(F(x), F(p)) < \epsilon$. Choose $M \in \mathbb{Z}_+$ large enough so that $1/M < \epsilon/2$. Because the f_n are all continuous, we may choose for each $n \leq M$ an open set $p \in U_n$ such that $f_n(U_n) \subseteq B_{\epsilon/2}(f_n(p))$. Set $U = U_1 \cap .... \cap U_M$. If $x \in U$, then

$$|f_n(x) - f_n(p)| < \epsilon/2 \qquad \begin{cases} \text{if } n \leq M \text{ because } x \in U \subseteq U_n \\ \text{if } n > M \text{ because } f(X) \subset [0, 1/M] \subset [0, \epsilon/2]. \end{cases}$$

Consequently $\rho(F(x), F(p)) = \sup\{|f_n(x) - f_n(p)|\} \leq \epsilon/2 < \epsilon$.

Step 4: $F^{-1} : F(X) \to X$ is continuous.

Proof. It suffices to show that for each $U \subset X$ open and $p \in U$ there exists an open set $V \subset [0, 1]^{\mathbb{Z}_+}$ such that

$$F(p) \in V \cap F(Z) \subseteq F(U).$$

Choose $n \in \mathbb{Z}_+$ so that $f_n(p) > 0$ and $f_n(U^c) = 0$. Set

$$V := \pi_n^{-1}((0, 1])$$

where $\pi_n : [0, 1]^{\mathbb{Z}_+} \to [0, 1]$ is the nth projection map. Then V is open in the product topology, hence also open in the uniform topology. Furthermore

$$p \in F^{-1}(V) = f_n^{-1}((0, 1]) \subseteq U$$

so $f(p) \in V \cap F(X) \subseteq f(U)$ □

We have shown that X embeds as a subspace of a metric space, hence is metrizable. □

7.3 Tychonoff's Theorem

Our goal in this section is to prove

Theorem 7.6 (Tychonoff's Theorem)**.** *Every product of compact spaces is compact.*

The proof will require some groundwork. First, we introduce a new characterization of compactness.

Definition 49. A collection $\mathcal{A}$ of subsets of X has the **finite intersection property** or **f.i.p.** if every finite collection $\{A_1, ..., A_n\} \in \mathcal{A}$ satisfies

$$A_1 \cap ... \cap A_n \neq \emptyset.$$

Proposition 7.7. *A space X is compact if and only if for every collection $\mathcal{A}$ of* closed *subsets possessing the f.i.p., the intersection of all sets $\bigcap_{A\in\mathcal{A}} A$ is nonempty.*

Proof. Let X be a compact set and $\mathcal{A}$ a collection of closed sets. Let $\tilde{\mathcal{A}} = \{A^c | A \in \mathcal{A}\}$ be the complementary collection of open sets. Using de Morgan's law

$$(\cap_n A_n)^c = \cup_n A_n^c$$

we see that $\mathcal{A}$ has f.i.p. if and only if $\tilde{\mathcal{A}}$ possesses no finite subcovers. Since X is compact, this implies that $\tilde{\mathcal{A}}$ is not a covering, which again by de Morgan's law implies that the intersection of all elements in $\mathcal{A}$ is non-empty. The converse follows by similar reasoning. □

Lemma 7.8. *Let $\mathcal{A}$ be a collection of subsets of X (not necessarily closed) possessing the f.i.p.. Then there exists a larger collection $\mathcal{D} \supset \mathcal{A}$ possessing the f.i.p. which is maximal in the sense that any strictly larger collection does not have f.i.p..*

Proof. Consider the set

$$\Lambda := \{\mathcal{E} \mid \mathcal{E} \text{ is a collection of subsets of } X \text{ possessing f.i.p. and } \mathcal{E} \supseteq \mathcal{A}\}$$

partially ordered by inclusion. We are seeking a maximal element of Λ and will use Zorn's Lemma to prove its existence.

It remains to show that Λ satisfies the hypotheses of Zorn's Lemma. Let $S \subset \Lambda$ be a linearly ordered subset, i.e. if $\mathcal{E}_1, \mathcal{E}_2 \in S$ then either $\mathcal{E}_1 \subseteq \mathcal{E}_2$ or $\mathcal{E}_2 \subseteq \mathcal{E}_1$.

Claim: $\mathcal{E} = \bigcup_{\mathcal{E}_i \in S} \mathcal{E}_i$ is an upperbound of S in Λ.

Proof. Clearly as a collection of sets $\mathcal{E}$ contains every collection in S. We need only show that $\mathcal{E}$ possesses the f.i.p.. Choose any finite collection of sets $A_1, ..., A_n \in \mathcal{E}$. For each $1 \leq k \leq n$ there exists $\mathcal{E}_k \in S$ such that $A_k \in \mathcal{E}_k$. Amongst these there is one, say $\mathcal{E}_i$ which contains all the other $\mathcal{E}_k$ (any finite linearly ordered set contains an upperbound). Thus $A_1, ..., A_n \in \mathcal{E}_i$ and since $\mathcal{E}_i$ possesses f.i.p., we get $A_1 \cap ... \cap A_n \neq \emptyset$. Thus $\mathcal{E}$ possesses f.i.p.. □

□

Example 78. Suppose that $\mathcal{A} = \{\{p\}\}$. Then there is a unique $\mathcal{D}$ equal to the collection of all subsets containing p.

Corollary 7.9. *X is compact if and only if every* <u>*maximal*</u> *collection of subsets $\mathcal{D}$ possessing the f.i.p. satisfies*

$$\bigcap_{A\in\mathcal{D}} \bar{A} \neq \emptyset.$$

Proof. If X is compact and $\mathcal{D}$ possesses f.i.p. then $\{\bar{A} | A \in \mathcal{D}\}$ also possesses f.i.p. so $\bigcap_{A\in\mathcal{D}} \bar{A} \neq \emptyset$.

On the other hand, if $\mathcal{A}$ is a collection of closed sets possessing f.i.p. and $\mathcal{D} \supset \mathcal{A}$ is a maximal collection possessing f.i.p. then

$$\bigcap_{A \in \mathcal{A}} A \supset \bigcap_{A \in \mathcal{D}} \bar{A} \neq \emptyset$$

implies that X is compact. □

Lemma 7.10. *Let $\mathcal{D}$ be a collection of subsets of X which is maximal with respect to the f.i.p.. Then*

(a) If $A \in \mathcal{D}$ and $B \supset A$ then $B \in \mathcal{D}$.

(b) $\mathcal{D}$ is closed under finite intersections.

(c) If $B \cap A \neq \emptyset$ for every $A \in \mathcal{D}$, then $B \in \mathcal{D}$.

Proof. (a)Closure under unions is clear. Let $\{A_1, ..., A_n\} \subseteq \mathcal{D}$ and let $A := A_1 \cap ... \cap A_n$. If $B_1, ..., B_m \in \mathcal{D}$ then

$$A \cap B_1 \cap ... \cap B_m = A_1 \cap ... \cap A_n \cap B_1 ... \cap B_m \neq \emptyset$$

so $\mathcal{D} \cup \{A\}$ possesses f.i.p.. By maximality, $A \in \mathcal{D}$.

(b) Let $A_1, ..., A_n \in \mathcal{D}$. Then

$$B \cap A_1 \cap ... \cap A_n = B \cap A \neq \emptyset,$$

so $B \in \mathcal{D}$. □

Proof of Tychonoff's Theorem. Let $\{X_\alpha\}_{\alpha \in I}$ be a collection of compact spaces. We want to show that the product space

$$X := \prod_{\alpha \in I} X_\alpha$$

is compact. Let $\mathcal{D}$ be a maximal collection of subsets of X possessing the f.i.p.. Our goal is to show that $\bigcap_{A \in \mathcal{D}} \bar{A} \neq \emptyset$.

Denote by $\pi_\alpha : X \to X_\alpha$ the projection map. Consider the collection of subsets $\{\pi_\alpha(A) \mid A \in \mathcal{D}\}$ of subsets of X_α. This collection possesses the finite intersection property because for any $\{A_1, ..., A_n\} \subset \mathcal{D}$,

$$\bigcap_{i=1}^{n} \pi_\alpha(A_i) \supseteq \pi_\alpha(\bigcap_{i=1}^{n} A_i) \neq \emptyset$$

Since X_α is compact, there exists a point

$$p_\alpha \in \bigcap_{A \in \mathcal{D}} \overline{\pi_\alpha(A)} \neq \emptyset.$$

□

Claim: The point $p = (p_\alpha)_{\alpha \in I} \in \bar{A}$ for all $A \in \mathcal{D}$.

Proof. The claim holds iff every open neighbourhood of $p \in U \subseteq X$ intersects every $A \in \mathcal{D}$ iff every open neighbourhood of p is contained in $\mathcal{D}$ by Lemma 7.10. Also by Lemma 7.10, $\mathcal{D}$ is closed under unions and finite intersections, so it is enough to show that every sub-basic open nbhd of p is in $\mathcal{D}$.

Recall that the sub-basic open sets in X are those of the form $\pi_\alpha^{-1}(U_\alpha)$ where U_α is open in X_α. Now $\pi^{-1}(U_\alpha)$ contains p if and only if $p_\alpha \in U_\alpha$. Since $p_\alpha \in \overline{\pi_\alpha(A)}$ for all $A \in \mathcal{D}$, we conclude that

$$U_\alpha \cap \overline{\pi_\alpha(A)} \neq \emptyset \Rightarrow U_\alpha \cap \pi_\alpha(A) \neq \emptyset \text{ } (U_\alpha \text{ is open}) \Rightarrow \pi^{-1}(U_\alpha) \cap A \neq \emptyset.$$

which implies by part (b) Lemma 7.10 that $\pi^{-1}(U_\alpha) \in \mathcal{D}$, completing the proof. □